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Differential graded versus simplicial categories ☆

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ABSTRACT

We establish a connection between differential graded and simplicial categories by constructing a three-step zig-zag of Quillen adjunctions relating the homotopy theories of the two. In an intermediate step, we extend the Dold–Kan correspondence to a Quillen equivalence between categories enriched over non-negatively graded complexes and categories enriched over simplicial modules. As an application, we obtain a simple calculation of Simpson's homotopy fiber, which is known to be a key step in the construction of a moduli stack of perfect complexes on a smooth projective variety.

0. Introduction

Dg categories: A *differential graded* (= *dg*) *category* is a category enriched in the category of complexes of modules over some commutative base ring k. Dg categories provide a framework for 'homological geometry' and 'non-commutative algebraic geometry' in the sense of Bondal, Drinfeld, Kapranov, Kontsevich, Toën, Van den Bergh, ... [5,6,9,10,19,20,33]. They are considered as enriched derived categories of quasi-coherent sheaves on a hypothetical non-commutative space; see Keller's ICM address [18]. Dg categories enhance triangulated categories and so they should be considered only up to the notion of *quasi-equivalence*, a hybrid notion involving quasi-isomorphisms and categorical equivalences; see Definition 2.13. For example, one can associate to a smooth projective algebraic variety X a *dg model* $\mathcal{D}^b_{dg}(\mathrm{coh}(X))$, i.e. a dg category well defined up to quasi-equivalence, whose triangulated category $\mathsf{H}_0(\mathcal{D}^b_{dg}(\mathrm{coh}(X)))$ (obtained by applying the zero homology group functor in each Hom-complex) is equivalent to the bounded derived category of quasi-coherent complexes on X. In [32], we constructed a homotopy theory of dg categories with respect to the class of quasi-equivalences; see Theorem 2.21. This theory enabled several developments such as: the creation by Toën of a derived Morita theory [33], the construction of a category of 'non-commutative motives' [31], and the first conceptual characterization [31] of Quillen–Waldhausen's K-theory [22,36] since its definition in the early 70's.

Simplicial categories: A *simplicial category* is a category enriched in the category of simplicial sets. Simplicial categories provide a framework for 'homotopy theories' and for 'higher topos' in the sense of Joyal, Lurie, Rezk, Toën, ... [16,23,27,34]. They are considered as simple models of $(\infty, 1)$-categories, i.e. ∞-categories whose i-morphisms are invertible for $i \geqslant 2$; see Bergner's survey [3]. As in the differential graded case, simplicial categories should only be considered up to the notion of *Dwyer–Kan equivalence* (see Dwyer and Kan [11,12]), a mixture of weak equivalences and categorical equivalences; see Definition 5.5. In [2], Bergner constructed a homotopy theory of simplicial categories with respect to the class of Dwyer–Kan equivalences. This theory is used nowadays in homotopy theory (see Bergner [4]), derived algebraic geometry (see Lurie [24] and Toën and Vezzosi [35]), and other areas of mathematics.

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Results: Even though the motivation for the development of differential graded and simplicial categories was different, the corresponding homotopy theories are formally similar and a 'dictionary' between the two should be developed. In this paper we establish a 'bridge' between these homotopy theories as follows:

In Section 3, starting from dg categories, we construct a Quillen model structure on the category $\mathsf{dgcat}_{\geqslant 0}$ (see Definition 3.1) of *connective dg categories*, i.e. categories enriched over non-negatively graded complexes; see Theorem 3.10. This model structure has recently been used by Moriya to prove a 'generalized Sullivan–de Rham equivalence'; see details in [26].

In Section 4, we consider categories enriched over simplicial k-modules. Let us denote by $s\mathbf{Mod}$-Cat the category of such objects. Inspired by Shipley–Schwede's work [28] on connective dg algebras (i.e. connective dg categories with a single object), we describe explicitly the left adjoint of the classical normalization functor $N : s\mathbf{Mod}\text{-Cat} \to \mathsf{dgcat}_{\geqslant 0}$; see Proposition 4.10. Then, we develop a functorial path object in $s\mathbf{Mod}$-Cat (see Definition 4.20), whose construction is of independent interest. Using this functorial path object, we lift the Quillen model structure from $\mathsf{dgcat}_{\geqslant 0}$ to $s\mathbf{Mod}$-Cat (see Theorem 4.16) and prove the following 'extended Dold–Kan equivalence':

Theorem 0.1. *(Theorem 4.30) There is a Quillen equivalence*

$$\begin{tikzcd} s\mathbf{Mod}\text{-Cat} \arrow[d, shift left, "N"] \\ \mathsf{dgcat}_{\geqslant 0} \arrow[u, shift left, "L"] \end{tikzcd}$$

between categories enriched over non-negatively graded complexes and categories enriched over simplicial k-modules.

The novelty in this 'extended Dold–Kan equivalence' is that, in contrast to connective dg algebras (see Schwede and Shipley [28, Theorem 1.1(3)]), the correct notion of equivalence (as the above example shows) is not defined by simply forgetting the multiplicative structure. It is a hybrid notion which involves weak equivalences and categorical equivalences (see Definitions 3.5 and 4.18), making not only the arguments different but also more involved.

In Section 5, using the k-linearization functor $k(-)$ between simplicial sets and simplicial k-modules, we construct a Quillen adjunction between simplicial categories and $s\mathbf{Mod}$-Cat; see Theorem 5.8.

Let us now sum up the construction of our three-step zig-zag of Quillen adjunctions:

$$\begin{tikzcd} s\mathbf{Set}\text{-Cat} \arrow[d, shift right, "k(-)"'] \\ s\mathbf{Mod}\text{-Cat} \arrow[u, shift right, "U"'] \arrow[d, shift left, "N"] \\ \mathsf{dgcat}_{\geqslant 0} \arrow[u, shift left, "L"] \arrow[d, hook, shift right, "i"'] \\ \mathsf{dgcat}. \arrow[u, shift right, "\tau_{\geqslant 0}"'] \end{tikzcd}$$

By Theorem 0.1, the adjunction (L, N) is a Quillen equivalence and so both derived functors between the homotopy categories

$$\mathbb{L}L : Ho(\mathsf{dgcat}) \longrightarrow Ho(s\mathbf{Mod}\text{-Cat}) \qquad \mathbb{R}N : Ho(s\mathbf{Mod}\text{-Cat}) \longrightarrow Ho(\mathsf{dgcat}_{\geqslant 0})$$

commute with homotopy limits and colimits. This implies that the composed functor

$$\Phi := U \circ \mathbb{L}L \circ \tau_{\geqslant 0} : Ho(\mathsf{dgcat}) \longrightarrow Ho(s\mathbf{Set}\text{-Cat})$$

preserves homotopy limits and the composed functor

$$\Psi := i \circ \mathbb{R}N \circ \mathbb{L}k(-) : Ho(s\mathbf{Set}\text{-Cat}) \longrightarrow Ho(\mathsf{dgcat})$$

preserves homotopy colimits. Hence, we obtain well-defined (exact) functors Φ and Ψ, which allow us to go back and forward between differential graded and simplicial categories. In particular, every construction/problem in $Ho(\mathsf{dgcat})$ involving homotopy limits, can be performed/solved in $Ho(s\mathbf{Set}\text{-Cat})$ and vice-versa. A consequence of our results is the following.

Corollary 0.2.

(1) *Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor and b an object of $\mathcal{B}$. We denote by $\mathbf{HFib}(F)/b$ the homotopy fiber of F over b. Then, we have an isomorphism*

$$\Phi\big(\mathbf{HFib}(F)/b\big) \xrightarrow{\sim} \mathbf{HFib}\big(\Phi(F)\big)/b$$

in the homotopy category $Ho(s\mathbf{Set}\text{-Cat})$.

(2) *Let $G : C \to D$ be a simplicial functor. We denote by* $\mathbf{HCof}(G)$ *the homotopy cofiber of G, i.e. the homotopy cobase change along the map to the terminal object. Then, we have an isomorphism*

$$\mathbf{HCof}\big(\Psi(G)\big) \xrightarrow{\sim} \Psi\big(\mathbf{HCof}(G)\big)$$

in the homotopy category $Ho(\mathsf{dgcat})$.

The connection established in this paper between differential graded and simplicial categories is becoming more and more important in different branches of mathematics such as non-abelian Hodge theory. Given a smooth projective variety X, Simpson constructed in [30] a moduli stack $\mathcal{M}_{Hod}(X, Perf)$ of 'perfect complexes with integrable connection' on X. His main theorem asserts that $\mathcal{M}_{Hod}(X, Perf)$ is locally geometric; see [30, Theorems 6.13 and 7.1]. A key step in his proof is the calculation of the homotopy fiber of the projection functor

$$\mathcal{M}_{Hod}(X, Perf) \longrightarrow Perf(X).$$

This corresponds to the *calculation* of the homotopy fiber of a certain dg functor from weak to perfect complexes under Φ. Simpson's homotopy fiber can be obtained by an application of Corollary 0.2 (see [30, §2.3]), which results in a simpler and more enlightening calculation.

1. Preliminaries

1.1. Notations

Throughout the paper k will denote commutative ring with unit.

Let $Ch(k)$ be the category of (unbounded) complexes of k-modules; see Hovey [14, §2.3]. We will use homological notation, i.e. the differential decreases the degree. The tensor product of complexes of k-modules will be denoted by $\otimes$. Given a complex $M_\bullet$ (with differential d) and an integer n, its *n-th suspension* $M_\bullet[n]$ is the complex given by $M_\bullet[n]_p = M_{p-n}$, $p \in \mathbb{Z}$, with differential $(-1)^n d$. The category $Ch(k)$ is a symmetric monoidal model category (see [14, Definition 4.2.6]), where one uses the projective model structure for which weak equivalences are quasi-isomorphisms and fibrations are surjections; see [14, Proposition 4.2.13].

Let $Ch_{\geqslant 0}(k) \subset Ch(k)$ be the symmetric monoidal full subcategory of *non-negatively graded complexes*, i.e. complexes $M_\bullet$ such that $M_n = 0$ for all $n < 0$. The category $Ch_{\geqslant 0}(k)$ is a symmetric monoidal model category, where one uses the projective model structure for which weak equivalences are quasi-isomorphisms and fibrations are the maps which are surjective in positive degrees; see Quillen [21, §II-4, Remark 5] or Goerss and Jardine [15, §III after Corollary 2.12]. We will denote by

$$\underline{Ch_{\geqslant 0}(k)}(-,-) : \big(Ch_{\geqslant 0}(k)\big)^{\mathrm{op}} \times Ch_{\geqslant 0}(k) \longrightarrow Ch_{\geqslant 0}(k)$$

the internal Hom-functor adjoint to the tensor product $\otimes$. Given non-negatively graded complexes $M_\bullet$, $N_\bullet$ and $P_\bullet$, we have the usual adjunction rule

$$\mathrm{Hom}_{Ch_{\geqslant 0}(k)}(M_\bullet \otimes N_\bullet, P_\bullet) \simeq \mathrm{Hom}_{Ch_{\geqslant 0}(k)}\big(M_\bullet, \underline{Ch_{\geqslant 0}(k)}(N_\bullet, P_\bullet)\big).$$

Let $s\mathbf{Set}$ be the category of simplicial sets; see Goerss and Jardine [15, §I]. The category $s\mathbf{Set}$ is a symmetric monoidal model category with weak equivalences the maps whose realization are weak equivalences of topological spaces. The fibrations are the Kan-fibrations and the cofibrations are the inclusion maps; see [15, Proposition 4.2.8].

Let $s\mathbf{Mod}$ be the category of simplicial k-modules; see Goerss and Jardine [15, §III-2]. We will denote by $\wedge$ the levelwise tensor product of simplicial k-modules. Note that every simplicial k-module has an underlying simplicial set. The category $s\mathbf{Mod}$ is a symmetric monoidal model category; see [15, Theorem 2.8-III]. The weak equivalences are the maps whose underlying maps of simplicial sets are weak equivalences. Similarly, the fibrations are the maps whose underlying maps of simplicial sets are Kan-fibrations. We will denote by

$$\underline{s\mathbf{Mod}}(-,-) : (s\mathbf{Mod})^{\mathrm{op}} \times s\mathbf{Mod} \longrightarrow s\mathbf{Mod}$$

the internal Hom-functor adjoint to the levelwise tensor product $\wedge$. Given simplicial k-modules A, B and C, we have the usual adjunction rule

$$\mathrm{Hom}_{s\mathbf{Mod}}(A \wedge B, C) \simeq \mathrm{Hom}_{s\mathbf{Mod}}\big(A, \underline{s\mathbf{Mod}}(B, C)\big).$$

Finally, the adjunctions will be displayed vertically, with the left adjoint on the left-hand side and the right adjoint on the right-hand side.

1.2. Monoidal functors

(See Schwede and Shipley [28, §3.2].) Let $(\mathcal{C}, -\otimes-, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, -\wedge-, \mathbb{I}_{\mathcal{D}})$ be two symmetric monoidal categories. Recall from [28, Definition 3.3] that a *lax monoidal functor* is a functor $R : \mathcal{C} \to \mathcal{D}$ equipped with a morphism $\nu : \mathbb{I}_{\mathcal{D}} \to R(\mathbb{I}_{\mathcal{C}})$ and morphisms

$$\varphi_{X,Y} : RX \wedge RY \longrightarrow R(X \otimes Y) \quad X, Y \in \mathcal{C},$$

which are coherently associative and unital (see the commutative diagrams 6.27 and 6.28 in Borceaux [7]). A lax monoidal functor is called *strong monoidal* if the morphisms ν and $\varphi_{X,Y}$ are isomorphisms.

Now, suppose that $R : \mathcal{C} \to \mathcal{D}$ is a lax monoidal functor with monoidal structure morphisms ν and $\varphi_{X,Y}$. If R has a left adjoint functor $\lambda : \mathcal{D} \to \mathcal{C}$, we have the morphism $\nu^\sharp : \lambda(\mathbb{I}_\mathcal{D}) \to \mathbb{I}_\mathcal{C}$ adjoint to ν. Moreover, we have morphisms

$$\tilde{\varphi} : \lambda(A \wedge B) \longrightarrow \lambda A \otimes \lambda B \quad A, B \in \mathcal{D},$$

adjoint to the composition

$$A \wedge B \xrightarrow{\eta_A \wedge \eta_B} R\lambda A \wedge R\lambda B \xrightarrow{\varphi_{\lambda A, \lambda B}} R(\lambda A \otimes \lambda B) \quad A, B \in \mathcal{D},$$

where η denotes the unit of the adjunction. In this situation, we say that the morphisms $\nu^\sharp$ and $\tilde{\varphi}$ endow λ with a *comonoidal structure*. The functor λ equipped with these morphisms is called a *comonoidal functor*.

2. Dg categories

In this section, we will review and develop all the aspects of the (homotopy) theory of differential graded (= dg) categories which are used throughout the paper. This will provide us with the occasion to fix some notation. For a survey paper on dg categories, we invite the reader to consult Keller's ICM address [18].

Definition 2.1. A *dg category* $\mathcal{A}$ is a $Ch(k)$-category; see Borceaux [7, Definition 6.2.1]. Recall that this consists of the following data:

- a set of objects $\mathrm{obj}(\mathcal{A})$ (usually denoted by $\mathcal{A}$ itself);
- for each pair of objects (x, y) in $\mathcal{A}$, a complex of k-modules $\mathcal{A}(x, y)$;
- for each triple of objects (x, y, z) in $\mathcal{A}$, a composition morphism in $Ch(k)$

  $$\mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \longrightarrow \mathcal{A}(x, z),$$

  satisfying the usual associativity condition;
- for each object x in $\mathcal{A}$, an element $\mathbf{1}_x \in \mathcal{A}(x, x)$, satisfying the usual unit condition with respect to the above composition.

Example 2.2. (Keller [18, §2.2]) The category $Ch(k)$ of complexes of k-modules is naturally a dg category: given two complexes $M_\bullet$ and $N_\bullet$, and an integer n, the k-module $Ch(k)(M_\bullet, N_\bullet)_n$ is formed by families $f = (f_p)$ of morphisms of k-modules:

$$f_p : M_p \longrightarrow N_{p+n} \quad p \in \mathbb{Z}.$$

The complex $Ch_{\mathrm{dg}}(k)(M_\bullet, N_\bullet)$ is the $\mathbb{Z}$-graded k-module with components the k-modules $Ch(k)(M_\bullet, N_\bullet)_n$, $n \in \mathbb{Z}$. Its differential is given by the commutator

$$d(f) = (d_{N_\bullet} \circ f) - (-1)^{\deg(f)}(f \circ d_{M_\bullet}).$$

We denote by $Ch_{\mathrm{dg}}(k)$ the dg category obtained in this way.

Definition 2.3. A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a $Ch(k)$-functor; see Borceaux [7, Definition 6.2.3]. Recall that this consists of the following data:

- a map of sets $F : \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$;
- for each pair of objects (x, y) in $\mathcal{A}$, a morphism in $Ch(k)$

  $$F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy),$$

  satisfying the usual unit and associativity conditions.

Notation 2.4. We denote by dgcat the category of dg categories.

Definition 2.5. Let $\mathcal{A}$ be a dg category.

- The *opposite dg category* $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$ has the same objects as $\mathcal{A}$ and complexes of morphisms given by $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$.
- The category $\mathsf{Z}_0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by $\mathsf{Z}_0(\mathcal{A})(x, y) := Z_0(\mathcal{A}(x, y))$, where Z_0 is the kernel of the differential map $d : \mathcal{A}(x, y)_0 \to \mathcal{A}(x, y)_{-1}$.
- The category $\mathsf{H}_0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by $\mathsf{H}_0(\mathcal{A})(x, y) := H_0(\mathcal{A}(x, y))$, where H_0 is the 0-th homology group.

We obtain well-defined functors:

$$\mathsf{Z}_0(-), \mathsf{H}_0(-) : \mathsf{dgcat} \longrightarrow \mathsf{Cat},$$

with values in the category of categories.

Definition 2.6. Let $\mathcal{A}$ be a dg category.

- A morphism $s : x \to y$ in $\mathsf{Z}_0(\mathcal{A})$ is a *homotopy equivalence* if it becomes invertible in $\mathsf{H}_0(\mathcal{A})$.
- An object x in $\mathcal{A}$ is called *contractible* if the dg algebra $\mathcal{A}(x,x)$ is acyclic, or equivalently if there exists an element $h \in \mathcal{A}(x,x)_1$ such that $d(h) = \mathbf{1}_x$; see Definition 2.1. Such an element h is called a *contraction* of x.

Definition 2.7. Let $F, G : \mathcal{A} \to \mathcal{B}$ be two dg functors. A *morphism* $\alpha : F \to G$ *of dg functors* is a $Ch(k)$-natural transformation; see Borceaux [7, Definition 6.2.4]. Recall that this consists of the following data:

- for each object x in $\mathcal{A}$, an element $\alpha_x \in Z_0(\mathcal{B}(Fx, Gx))$, satisfying the usual commutativity condition.

Notation 2.8. Given dg categories $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ the category of dg functors. Its objects are the dg functors (Definition 2.3) and its morphisms are the morphisms of dg functors (Definition 2.7). Recall from Keller [18, §2.3] that (as in Example 2.2) $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ is naturally a dg category: given dg functors $F, G : \mathcal{A} \to \mathcal{B}$ and an integer n, the k-module $\mathsf{Fun}(\mathcal{A}, \mathcal{B})(F, G)_n$ is formed by families of elements

$$\alpha_x \in \mathcal{B}(Fx, Gx)_n \quad x \in \mathcal{A}$$

such that $(Gf)(\alpha_x) = (\alpha_x)(Ff)$ for all $f \in \mathcal{A}(y, y')$, $y, y' \in \mathcal{A}$. The differential is induced by that of $\mathcal{B}(Fx, Gx)$. We denote by $\mathsf{Fun}_{\mathsf{dg}}(\mathcal{A}, \mathcal{B})$ the associated dg category. Note that we have an equivalence of categories $\mathsf{Z}_0\mathsf{Fun}_{\mathsf{dg}}(\mathcal{A}, \mathcal{B}) \simeq \mathsf{Fun}(\mathcal{A}, \mathcal{B})$.

2.1. Dg modules

Definition 2.9. Let $\mathcal{A}$ be a dg category. A *right dg $\mathcal{A}$-module M* (or simply a $\mathcal{A}$-module) is a dg functor $M : \mathcal{A}^{\mathsf{op}} \to Ch_{\mathsf{dg}}(k)$ (see Example 2.2). A *morphism of $\mathcal{A}$-modules* is a morphism of dg functors (see Definition 2.7).

Notation 2.10. We denote by $\mathcal{C}(\mathcal{A})$ the category $\mathsf{Fun}(\mathcal{A}^{\mathsf{op}}, Ch_{\mathsf{dg}}(k))$ of $\mathcal{A}$-modules and by $\mathcal{C}_{\mathsf{dg}}(\mathcal{A})$ the associated dg category $\mathsf{Fun}_{\mathsf{dg}}(\mathcal{A}^{\mathsf{op}}, Ch_{\mathsf{dg}}(k))$; see Notation 2.8. Recall that we have an equivalence of categories $\mathsf{Z}_0\mathcal{C}_{\mathsf{dg}}(\mathcal{A}) \simeq \mathcal{C}(\mathcal{A})$.

Thanks to Keller [18, Theorem 3.2] the category $\mathcal{C}(\mathcal{A})$ carries the projective model structure, whose fibrations are the objectwise surjections and whose weak equivalences are the objectwise quasi-isomorphisms. We denote by $\mathcal{D}(\mathcal{A})$ the *derived category of* $\mathcal{A}$, i.e. the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. A dg functor $F : \mathcal{A} \to \mathcal{B}$ gives rise to *restriction/extension of scalars* adjunctions (see Toën [33, §3]):

$$\begin{tikzcd} \mathcal{C}(\mathcal{B}) \arrow[d, shift left, "F^*"] \\ \mathcal{C}(\mathcal{A}) \arrow[u, shift left, "F_!"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{D}(\mathcal{B}) \arrow[d, shift left, "F^*"] \\ \mathcal{D}(\mathcal{A}). \arrow[u, shift left, "\mathbb{L}F_!"] \end{tikzcd}$$

The functor F^* is the restriction of scalars functor. The functor $F_!$ is the extension of scalars functor and $\mathbb{L}F_!$ is its derived functor. The functors F^* and $F_!$ respect the differential graded structures of $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$, and so give rise to an adjunction:

$$\begin{tikzcd} \mathcal{C}_{\mathsf{dg}}(\mathcal{B}) \arrow[d, shift left, "F^*"] \\ \mathcal{C}_{\mathsf{dg}}(\mathcal{A}). \arrow[u, shift left, "F_!"] \end{tikzcd}$$

Definition 2.11. Let $\mathcal{A}$ be a dg category. The *Yoneda dg functor*

$$\widehat{(-)} : \mathcal{A} \longrightarrow \mathcal{C}_{\mathsf{dg}}(\mathcal{A}) \qquad x \mapsto \mathcal{A}(-, x) =: \hat{x}$$

sends an object $x \in \mathcal{A}$ to the $\mathcal{A}$-module $\mathcal{A}(-, x)$ *represented by* x.

Definition 2.12. (Keller [17, §2]) Let $\mathcal{A}$ be a dg category, x and y objects in $\mathcal{A}$, and $s \in \mathsf{Z}_0(\mathcal{A})(x, y)$ (see Definition 2.5). The *cone of* $\hat{s}$, which will be denoted by $\mathsf{cone}(\hat{s})$, is the $\mathcal{A}$-module defined as follows: its underlying $\mathbb{Z}$-graded $\mathcal{A}$-module structure is the one of the $\mathcal{A}$-module $\hat{y} \oplus \hat{x}[1]$, where $\hat{x}[1]$ denotes the suspension of $\hat{x}$ in $\mathcal{C}(\mathcal{A})$. Its differential is given by the following matrix:

$$\begin{bmatrix} d_{\hat{y}} & \hat{s} \\ 0 & -d_{\hat{x}} \end{bmatrix}.$$

2.2. Quillen model structure

Definition 2.13. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a *quasi-equivalence* if:

(QE1) for all objects $x, y \in \mathcal{A}$, the morphism in $Ch(k)$

$$F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy)$$

is a quasi-isomorphism and

(QE2) the induced functor (see Definition 2.5)

$$\mathsf{H}_0(F) : \mathsf{H}_0(\mathcal{A}) \longrightarrow \mathsf{H}_0(\mathcal{B})$$

is an equivalence of categories.

Notation 2.14. We denote by $\mathcal{Q}e$ the class of quasi-equivalences in dgcat.

Definition 2.15. (Drinfeld [9, Definition 3.7.1]) Let $\mathcal{K}$ be the dg category with two objects 1 and 2, and complexes of morphisms generated by the morphisms $f \in \mathcal{K}(1,2)_0$, $g \in \mathcal{K}(2,1)_0$, $r_1 \in \mathcal{K}(1,1)_1$, $r_2 \in \mathcal{K}(2,2)_1$, and $r_{12} \in \mathcal{K}(1,2)_2$:

$$\begin{tikzcd} 1 \arrow[loop left, "r_1"] \arrow[r, bend left=20, "f"] \arrow[r, bend left=70, "r_{12}"] & 2 \arrow[l, bend left=20, "g"] \arrow[loop right, "r_2"] \end{tikzcd}$$

subject to the relations $d(f) = d(g) = 0$, $d(r_1) = g \circ f - \mathbf{1}_1$, $d(r_2) = f \circ g - \mathbf{1}_2$ and $d(r_{12}) = f \circ r_1 - r_2 \circ f$.

Lemma 2.16. *Let $\mathcal{B}$ be a dg category. There is a bijection between the set of dg functors from $\mathcal{K}$ to $\mathcal{B}$ and the set of pairs (s, h), where s is a morphism in $\mathsf{Z}_0(\mathcal{B})$ and h is a contraction of the object* $\mathsf{cone}(\hat{s})$ *in $\mathcal{C}_{dg}(\mathcal{B})$; see Definitions* 2.12 *and* 2.6.

Proof. The dg category $\mathcal{K}$ is defined by generators and relations. Hence, the datum of a dg functor $F : \mathcal{K} \to \mathcal{B}$ corresponds to the datum of morphisms in $\mathcal{B}$: $s \in \mathcal{B}(x, y)_0$, $p \in \mathcal{B}(y, x)_0$, $r_x \in \mathcal{B}(x, x)_1$, $r_y \in \mathcal{B}(y, y)_1$, and $r_{xy} \in \mathcal{B}(x, y)_2$ subject to the relations $d(s) = 0$ and

$$d(p) = 0 \qquad d(r_x) = p \circ s - \mathbf{1}_x \qquad d(r_y) = s \circ p - \mathbf{1}_y \qquad d(r_{xy}) = s \circ r_x - r_y \circ s. \tag{2.1}$$

The morphisms s, p, r_x, r_y and r_{xy} in $\mathcal{B}$ are, respectively, the images of the morphisms f, g, r_1, r_2 and r_{12} in $\mathcal{K}$ under the dg functor F.

On the other hand, note that by definition a morphism s in $\mathsf{Z}_0(\mathcal{B})$ is a morphism $s \in \mathcal{B}(x, y)_0$ which satisfies the relation $d(s) = 0$. Note also that a degree one morphism h in $\mathcal{C}_{\mathsf{dg}}(\mathcal{B})(\mathsf{cone}(\hat{s}), \mathsf{cone}(\hat{s}))$ (see Notation 2.10) corresponds to a matrix

$$\begin{bmatrix} \widehat{r_y} & \widehat{r_{xy}} \\ \hat{p} & \widehat{r_x} \end{bmatrix},$$

with $p \in \mathcal{B}(y, x)_0$, $r_x \in \mathcal{B}(x, x)_1$, $r_y \in \mathcal{B}(y, y)_1$ and $r_{xy} \in \mathcal{B}(x, y)_2$ morphisms in $\mathcal{B}$. This morphism h is a contraction if and only if its differential equals the identity of the cone of $\hat{s}$. This corresponds to the following equation:

$$\begin{bmatrix} \widehat{r_y} & \widehat{r_{xy}} \\ \hat{p} & \widehat{r_x} \end{bmatrix} \begin{bmatrix} d_{\hat{y}} & \hat{s} \\ 0 & -d_{\hat{x}} \end{bmatrix} - (-1)^{-1} \begin{bmatrix} d_{\hat{y}} & \hat{s} \\ 0 & -d_{\hat{x}} \end{bmatrix} \begin{bmatrix} \widehat{r_y} & \widehat{r_{xy}} \\ \hat{p} & \widehat{r_x} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{1}_y} & 0 \\ 0 & \widehat{\mathbf{1}_x} \end{bmatrix}.$$

By performing this matrix multiplication and taking into account the fully-faithfull nature of the Yoneda dg functor, we recover the relations (2.1). The proof is then finished. $\square$

Definition 2.17.

(i) Let $\underline{k}$ be the dg category with one object 3, such that $\underline{k}(3,3) := k$ (in degree zero). Note that given a dg category $\mathcal{B}$, there is a bijection between the set of dg functors from $\underline{k}$ to $\mathcal{B}$ and the set of objects of $\mathcal{B}$.

(ii) For $n \in \mathbb{Z}$, let S_n be the complex $k[n]$ (with k concentrated in degree n) and let D_n be the mapping cone on the identity of S_{n-1}. We denote by $\mathcal{S}(n)$ the dg category with two objects 4 and 5 such that $\mathcal{S}(n)(4,4) = k$, $\mathcal{S}(n)(5,5) = k$, $\mathcal{S}(n)(5,4) = 0$, $\mathcal{S}(n)(4,5) = S_n$ and with composition given by multiplication.

(iii) We denote by $\mathcal{D}(n)$ the dg category with two objects 6 and 7 such that $\mathcal{D}(n)(6,6) = k$, $\mathcal{D}(n)(7,7) = k$, $\mathcal{D}(n)(7,6) = 0$, $\mathcal{D}(n)(6,7) = D_n$ and with composition given by multiplication. Let $\iota(n) : \mathcal{S}(n-1) \to \mathcal{D}(n)$ be the dg functor that sends 4 to 6, 5 to 7 and S_{n-1} to D_n by the identity on k in degree $n-1$:

$$\begin{tikzcd}
\mathcal{S}(n-1) \arrow[rr, "\iota(n)"] \arrow[d, equal] & & \mathcal{D}(n) \arrow[d, equal] \\
4 \arrow[loop above, "k"] \arrow[d, "S_{n-1}"'] \arrow[rr, mapsto] & \xrightarrow{\text{incl}} & 6 \arrow[loop above, "k"] \arrow[d, "D_n"] \\
5 \arrow[loop below, "k"] \arrow[rr, mapsto] & & 7 \arrow[loop below, "k"]
\end{tikzcd}
\quad \text{where} \quad
\begin{tikzcd}
S_{n-1} \arrow[r, "\text{incl}"] \arrow[d, equal] & D_n \arrow[d, equal] \\
\vdots & \vdots \\
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & k \arrow[d, "\text{id}"] \\
k \arrow[r, "\text{id}"] \arrow[d] & k \arrow[d] & (\text{degree } n-1) \\
0 \arrow[r] & 0 \\
\vdots & \vdots
\end{tikzcd}$$

Notation 2.18. Let I be the set consisting of the dg functors $\{\iota(n)\}_{n \in \mathbb{Z}}$ and the dg functor $\emptyset \to \underline{k}$ (where the empty dg category $\emptyset$ is the initial object in dgcat).

Definition 2.19. Let $\mathcal{B}_0$ be the dg category with two objects 8 and 9 such that $\mathcal{B}_0(8,8) = k$, $\mathcal{B}_0(9,9) = k$, $\mathcal{B}_0(4,9) = 0$, $\mathcal{B}_0(9,9) = 0$, and with composition given by multiplication. Let $\zeta(n) : \mathcal{B}_0 \to \mathcal{D}(n)$ be the dg functor that sends 8 to 6 and 9 to 7.

Notation 2.20. Let J be the set consisting of the dg functors $\{\zeta(n)\}_{n \in \mathbb{Z}}$ and the dg functor $\underline{k} \to \mathcal{K}$ that sends 3 to 1; see Definitions 2.15 and 2.17(i).

Theorem 2.21. *([32, Theorem 2.1]) Let $\mathcal{M}$ be the category* dgcat, *W the class $\mathcal{Q}e$ of quasi-equivalences, I the set of dg functors of Notation* 2.18*, and J the set of dg functors of Notation* 2.20*. Then the conditions of the recognition Theorem* A.3 *are satisfied, and so the category* dgcat *admits a cofibrantly generated Quillen model structure whose weak equivalences are the quasi-equivalences; see Definition* 2.13*.*

By Theorem 2.21, the fibrations in dgcat are the dg functors which have the right lifting property with respect to the dg functors of the set J. Let us now give an intrinsic characterization of the fibrations.

Proposition 2.22. *A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a fibration, with respect to the model structure of Theorem* 2.21*, if and only if:*

(F1) *for all objects $x, y \in \mathcal{A}$, the morphism in $Ch(k)$*

$$F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy)$$

is a surjection and

(F2) *given an object x in $\mathcal{A}$ and a homotopy equivalence $v : Fx \to y$ in $\mathcal{B}$ (see Definition* 2.6*), there exists a homotopy equivalence $u : x \to x'$ in $\mathcal{A}$, such that $Fu = v$.*

Proof. Note first that a dg functor satisfies condition (F1) if and only if it has the right lifting property with respect to the dg functors $\{\zeta(n)\}_{n \in \mathbb{Z}}$.

We now show that if a dg functor has the right lifting property with respect to the dg functor $\underline{k} \to \mathcal{K}$ of the set J, then it satisfies condition (F2). Let x be an object in $\mathcal{A}$ and $v : Fx \to y$ a homotopy equivalence in $\mathcal{B}$. Since v is a homotopy equivalence, the cone of $\hat{v}$ is a contractible object in $\mathcal{C}_{\mathsf{dg}}(\mathcal{B})$. Using Lemma 2.16, we can construct a dg functor $T : \mathcal{K} \to \mathcal{B}$ such that $T(f) = v$. We obtain then a solid commutative square:

$$\begin{tikzcd}
\underline{k} \arrow[r, "R"] \arrow[d] & \mathcal{A} \arrow[d, "F"] \\
\mathcal{K} \arrow[r, "T"'] \arrow[ur, dashed, "\overline{T}"] & \mathcal{B},
\end{tikzcd}$$

with $R(3) = x$. By hypothesis F has the right lifting property with respect to the dg functor $\underline{k} \to \mathcal{K}$, and so we obtain a dg functor $\overline{T}$ making the above diagram commute. Thanks to Lemma 2.16, the image of f under $\overline{T}$ is a homotopy equivalence $u : x \to x'$ in $\mathcal{A}$, and moreover $Fu = v$. This shows that every fibration in dgcat satisfies conditions (F1) and (F2).

Let us now prove the converse. Given a dg functor F satisfying conditions (F1) and (F2), we need to show that it has the right lifting property with respect to the dg functor $\underline{k} \to \mathcal{K}$ of the set J. Consider a commutative square in dgcat

$$\begin{tikzcd} \underline{k} \arrow[r, "R"] \arrow[d] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{K} \arrow[r, "T"'] & \mathcal{B}. \end{tikzcd}$$

By Definition 2.17(i) and Lemma 2.16 this commutative square corresponds to an object $x := R(3)$ in $\mathcal{A}$, a morphism $v : Fx \to y$ in $\mathsf{Z}_0(\mathcal{B})$ (which is the image of f under T), and a contraction of the cone of $\hat{v}$ in $\mathcal{C}_{\mathsf{dg}}(\mathcal{B})$. Since $\mathsf{cone}(\hat{v})$ is contractible, the morphism v is a homotopy equivalence and so by condition (F2) we obtain a homotopy equivalence $u : x \to x'$ such that $Fu = v$. The morphisms u and v are homotopy equivalences, and so the objects $\mathsf{cone}(\hat{u})$ and $\mathsf{cone}(\hat{v})$ are contractible. The extension of scalars dg functor (see Section 2.1)

$$F_! : \mathcal{C}_{\mathsf{dg}}(\mathcal{A}) \longrightarrow \mathcal{C}_{\mathsf{dg}}(\mathcal{B})$$

sends the cone of $\hat{u}$ to the cone of $\hat{v}$. Since F satisfies condition (F1), we obtain an induced surjective map

$$\mathcal{C}_{\mathsf{dg}}(\mathcal{A})\big(\mathsf{cone}(\hat{u}), \mathsf{cone}(\hat{u})\big) \longrightarrow \mathcal{C}_{\mathsf{dg}}(\mathcal{B})\big(\mathsf{cone}(\hat{v}), \mathsf{cone}(\hat{v})\big)$$

between acyclic dg algebras. Using Lemma 2.23, we can then lift the contraction of $\mathsf{cone}(\hat{v})$ (given by the dg functor T) to a contraction of $\mathsf{cone}(\hat{u})$. Finally, by Lemma 2.16 we obtain then a lift $\overline{T}$ of T making the diagram

$$\begin{tikzcd} \underline{k} \arrow[r, "R"] \arrow[d] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{K} \arrow[r, "T"'] \arrow[ur, dashed, "\overline{T}"] & \mathcal{B} \end{tikzcd}$$

commute. This shows that F has the right lifting property with respect to the dg functor $\underline{k} \to \mathcal{K}$ of the set J, and so the proof is finished. $\square$

Lemma 2.23. *Let $f : A \to B$ be a surjective map between acyclic dg algebras. Given an element $h \in B_1$, such that $d(h) = \mathbf{1}_B$, there exists an element $h' \in A_1$, such that $d(h') = \mathbf{1}_A$ and $f(h') = h$.*

Proof. Since f is a surjective map, we have a short exact sequence of complexes

$$0 \to K \hookrightarrow A \xrightarrow{f} B \to 0, \tag{2.2}$$

where K denotes the kernel of f. The dg algebras A and B are acyclic, and so the map f is a quasi-isomorphism. Therefore, the homology of complex K is trivial. Choose first a $w \in A_1$ such that $f(w) = h$. Consider the element $d(w) - \mathbf{1}_A$. Note that we have $d(d(w) - \mathbf{1}_A) = 0$ and $f(d(w) - \mathbf{1}_A) = 0$. The above short exact sequence (2.2) and the triviality of the homology of K imply that there is a $v \in A_1$ such that $d(v) = d(w) - \mathbf{1}_A$ and $f(v) = 0$. Finally, let $h' := w - v$. Then $d(h') = \mathbf{1}_A$ and $f(h') = h$ as required. $\square$

Lemma 2.24. *Every object in* dgcat *is fibrant with respect to the model structure of Theorem* 2.21.

Proof. The terminal object in dgcat (denoted by 0) is the dg category with a single object $*$ and trivial endomorphisms dg algebra. Therefore, given a dg category $\mathcal{A}$ the unique dg functor $P : \mathcal{A} \to 0$ satisfies condition (F1) of Proposition 2.22. Given an object $x \in \mathcal{A}$ and a homotopy equivalence $v : Px = * \to *$, the identity $\mathbf{1}_x : x \to x$ of x (see Definition 2.1) is a homotopy equivalence such that $P(\mathbf{1}_x) = v$. Therefore the functor F satisfies also the condition (F2) of Proposition 2.22, and so the proof is finished. $\square$

Proposition 2.25. *(Toën [33, Proposition 2.3]) Let $\mathcal{A}$ be a cofibrant object in* dgcat*. Then for all objects x and y in $\mathcal{A}$, the complex $\mathcal{A}(x, y)$ is cofibrant in $Ch(k)$ (see Section* 1.1*).*

2.3. Path object

Definition 2.26. Let $\mathcal{B}$ be a dg category. The dg category $P(\mathcal{B})$ is defined as follows: its objects are the homotopy equivalences

$$f : x \longrightarrow y$$

in $\mathcal{B}$ (see Definition 2.6). Its $\mathbb{Z}$-graded k-modules of morphisms are given by

$$P(\mathcal{B})\big(x \xrightarrow{f} y, w \xrightarrow{g} z\big) := \mathcal{B}(x,w) \oplus \mathcal{B}(y,z) \oplus \mathcal{B}(x,z)[1].$$

A homogeneous element of degree r is represented by a matrix

$$\begin{bmatrix} m_1 & h \\ 0 & m_2 \end{bmatrix},$$

with $m_1 \in \mathcal{B}(x,w)_r$, $m_2 \in \mathcal{B}(y,z)_r$ and $h \in \mathcal{B}(x,z)_{r+1}$. Under this notation, the differential is given by:

$$d\left(\begin{bmatrix} m_1 & h \\ 0 & m_2 \end{bmatrix}\right) := \begin{bmatrix} d(m_1) & d(h) + g \circ m_1 - (-1)^r (m_2 \circ f) \\ 0 & d(m_2) \end{bmatrix}.$$

Composition in $P(\mathcal{B})$ corresponds to matrix multiplication and the units correspond to the identity matrices.

We have an 'inclusion' dg functor

$$\gamma : \mathcal{B} \longrightarrow P(\mathcal{B}),$$

which sends an object $x \in \mathcal{B}$ to the identity $(\mathbf{1}_x : x = x)$ and a 'projection' dg functor

$$\pi_0 \times \pi_1 : P(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B},$$

which sends a homotopy equivalence $(f : x \to y)$ to (x, y). We obtain then the following commutative diagram in dgcat

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\Delta"] \arrow[dr, "\gamma"'] & & \mathcal{B} \times \mathcal{B} \\ & P(\mathcal{B}), \arrow[ur, "\pi_0 \times \pi_1"'] & \end{tikzcd}$$

where Δ is the diagonal dg functor.

Definition 2.27. (Hirschhorn [13, Definition 7.3.2(3)]) Let $\mathcal{M}$ be a Quillen model category and X an object of $\mathcal{M}$. A *path object* $P(X)$ for X is a factorization

$$\begin{tikzcd} X \arrow[rr, "\Delta"] \arrow[dr, "\gamma"', "\sim"] & & X \times X \\ & P(X) \arrow[ur, "\pi_0 \times \pi_1"'] & \end{tikzcd}$$

of the diagonal map (so that both compositions $\pi_0 \circ \gamma$ and $\pi_1 \circ \gamma$ are equal to the identity map), such that γ is a weak equivalence and $\pi_0 \times \pi_1$ is a fibration.

Proposition 2.28. *Let $\mathcal{B}$ be a dg category. The dg category $P(\mathcal{B})$ of Definition* 2.26 *is a path object for $\mathcal{B}$, with respect to the model structure of Theorem* 2.21.

Proof. We show first that the dg functor γ is a quasi-equivalence; see Definition 2.13. For all objects $x, y \in \mathcal{B}$, the dg functor γ induces a quasi-isomorphism in $Ch(k)$

$$\mathcal{B}(x,y) \xrightarrow{\sim} P(\mathcal{B})\big(\gamma(x), \gamma(y)\big),$$

and so it satisfies condition (QE1). Given an object $f : x \to y$ in $P(\mathcal{B})$, consider the following degree zero morphism in $P(\mathcal{B})$:

$$\begin{tikzcd} x \arrow[r, equal, "\mathbf{1}_x"] \arrow[d, equal, "\mathbf{1}_x"'] \arrow[dr, phantom, "h=0"] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

from $\gamma(x) := (\mathbf{1}_x : x = x)$ to $f : x \to y$. Since f is a homotopy equivalence in $\mathcal{B}$, the above degree zero morphism in $P(\mathcal{B})$ becomes invertible in $\mathsf{H}_0(P(\mathcal{B}))$. This shows that γ satisfies also condition (QE2). In sum, γ is a quasi-equivalence.

We now show that the dg functor $\pi_0 \times \pi_1$ is a fibration; see Proposition 2.22. By construction, the dg functor $\pi_0 \times \pi_1$ induces for all objects $f : x \to y$ and $g : w \to z$ in $P(\mathcal{B})$ a surjection:

$$\mathcal{B}(x,w) \oplus \mathcal{B}(y,z) \oplus \mathcal{B}(x,z)[1] \longrightarrow \mathcal{B}(x,w) \oplus \mathcal{B}(y,z).$$

Therefore it satisfies condition (F1). We now show that it satisfies condition (F2). We start by showing that contractions (see Definition 2.6) lift along $\pi_0 \times \pi_1$. Let $f : x \to y$ be an object in $P(\mathcal{B})$. Note that a contraction of f in $P(\mathcal{B})$ corresponds to morphisms $c_x \in \mathcal{B}(x,x)_1$, $c_y \in \mathcal{B}(y,y)_1$, and $c_{x,y} \in \mathcal{B}(x,y)_2$, subject to the relations $d(c_x) = \mathbf{1}_x$, $d(c_y) = \mathbf{1}_y$, and $d(c_{x,y}) = c_y \circ f + f \circ c_x$. Therefore, given a contraction (c_1, c_2) of (x, y) in $\mathcal{B} \times \mathcal{B}$, we can lift it by taking $c_x = c_1$, $c_y = c_2$ and $c_{x,y} = c_2 \circ f \circ c_1$. We are now ready to show condition (F2). Let $f : x \to y$ be an object in $P(\mathcal{B})$ and m a homotopy equivalence in $\mathcal{B} \times \mathcal{B}$ from $(\pi_0 \times \pi_1)(f) = (x, y)$ to (w, z). Note that the object $\mathsf{cone}(\hat{m})$ is contractible in $\mathcal{C}_{\mathsf{dg}}(\mathcal{B} \times \mathcal{B})$. Since the dg functor $\pi_0 \times \pi_1$ satisfies condition (F1), there exist an object $g : w \to z$ in $P(\mathcal{B})$ and a morphism $\underline{m}$ in $P(\mathcal{B})(x \xrightarrow{f} y, w \xrightarrow{g} z)$ such that $(\pi_0 \times \pi_1)(\underline{m}) = m$. Moreover, the extension of scalars dg functor (see Section 2.1)

$$(\pi_0 \times \pi_1)_! : \mathcal{C}_{\mathsf{dg}}\big(P(\mathcal{B})\big) \longrightarrow \mathcal{C}_{\mathsf{dg}}(\mathcal{B} \times \mathcal{B})$$

sends the cone of $\hat{\underline{m}}$ to the cone of $\hat{m}$. Since contractions lift along the dg functor $\pi_0 \times \pi_1$, they lift also along the dg functor $(\pi_0 \times \pi_1)_!$. Therefore, since the object $\mathsf{cone}(\hat{m})$ is contractible in $\mathcal{C}_{\mathsf{dg}}(\mathcal{B} \times \mathcal{B})$, the object $\mathsf{cone}(\hat{\underline{m}})$ is contractible in $\mathcal{C}_{\mathsf{dg}}(P(\mathcal{B}))$. In conclusion, the morphism $\underline{m}$ is a homotopy equivalence and so the dg functor $\pi_0 \times \pi_1$ satisfies condition (F2). The proof is then finished. □

3. Connective dg categories

In this section, we will construct a Quillen model structure on the category of *connective* dg categories; see Theorem 3.10.

Definition 3.1. A *connective dg category* is a $Ch_{\geqslant 0}(k)$-category (see Borceaux [7, Definition 6.2.1]) and a dg functor between connective dg categories is a $Ch_{\geqslant 0}(k)$-functor (see [7, Definition 6.2.3]).

Remark 3.2. Note that Definition 3.1 is no more than a combination of Definitions 2.1 and 2.3 in which $Ch(k)$ was replaced with $Ch_{\geqslant 0}(k)$.

Notation 3.3. We denote by $\mathsf{dgcat}_{\geqslant 0}$ the category of connective dg categories.

Remark 3.4. All the notions of Section 2 have a direct analogue in the context of connective dg categories. For example, given a connective dg category $\mathcal{A}$ we have well-defined categories $\mathsf{Z}_0(\mathcal{A})$ and $\mathsf{H}_0(\mathcal{A})$; see Definition 2.5. Since $Ch_{\geqslant 0}(k)$ is a full subcategory of $Ch(k)$, the category $\mathsf{dgcat}_{\geqslant 0}$ is a full subcategory of dgcat. Moreover, the inclusion functor

$$\mathsf{dgcat}_{\geqslant 0} \hookrightarrow \mathsf{dgcat}$$

preserves limits and colimits.

3.1. Quillen model structure

Definition 3.5. A dg functor $F : \mathcal{A} \to \mathcal{B}$ in $\mathsf{dgcat}_{\geqslant 0}$ is a *quasi-equivalence* if:

$(\mathrm{QE1})_{\geqslant 0}$ for all objects $x, y \in \mathcal{A}$, the morphism in $Ch_{\geqslant 0}(k)$

$$F(x,y) : \mathcal{A}(x,y) \longrightarrow \mathcal{B}(Fx, Fy)$$

is a quasi-isomorphism and

$(\mathrm{QE2})_{\geqslant 0}$ the induced functor (see Remark 3.4)

$$\mathsf{H}_0(F) : \mathsf{H}_0(\mathcal{A}) \longrightarrow \mathsf{H}_0(\mathcal{B})$$

is an equivalence of categories.

Notation 3.6. We denote by $\mathcal{Q}e_{\geqslant 0}$ the class of quasi-equivalences in $\mathsf{dgcat}_{\geqslant 0}$.

Remark 3.7. The class $\mathcal{Q}e_{\geqslant 0}$ consists of those quasi-equivalences in dgcat (see Definition 2.13) which belong to $\mathsf{dgcat}_{\geqslant 0}$, under the inclusion functor $\mathsf{dgcat}_{\geqslant 0} \hookrightarrow \mathsf{dgcat}$.

Definition 3.8. A dg functor $F : \mathcal{A} \to \mathcal{B}$ in $\mathsf{dgcat}_{\geqslant 0}$ is a *fibration* if:

$(\mathrm{F1})_{\geqslant 0}$ for all objects $x, y \in \mathcal{A}$, the morphism in $Ch_{\geqslant 0}(k)$

$$F(x,y) : \mathcal{A}(x,y) \longrightarrow \mathcal{B}(Fx, Fy)$$

is surjective in positive degrees and

$(F2)_{\geqslant 0}$ given an object x in $\mathcal{A}$ and a homotopy equivalence $v : Fx \to y$ in $\mathcal{B}$ (see Definition 2.6), there exists a homotopy equivalence $u : x \to x'$ in $\mathcal{A}$, such that $Fu = v$.

Definition 3.9. A dg functor $F : \mathcal{A} \to \mathcal{B}$ in $\mathsf{dgcat}_{\geqslant 0}$ is a *cofibration* if it has the left lifting property with respect to the dg functors in $\mathsf{dgcat}_{\geqslant 0}$ which are simultaneously quasi-equivalences (see Definition 3.5) and fibrations (see Definition 3.8).

Theorem 3.10. *The category* $\mathsf{dgcat}_{\geqslant 0}$ *endowed with the notions of weak equivalence, fibration, and cofibration of Definitions* 3.5, 3.8, *and* 3.9 *is a Quillen model category. Moreover, this model structure is cofibrantly generated*; *see Hirschhorn* [13, Definition 11.1.2].

Proof of Theorem 3.10. Recall from Notation 2.18 that we have dg functors $\{\iota(n)\}_{n\in\mathbb{Z}}$ and $\emptyset \to \underline{k}$, and from Notation 2.20 that we have dg functors $\{\zeta(n)\}_{n\in\mathbb{Z}}$ and $\underline{k} \to \mathcal{K}$. Let $N : \mathcal{B}_0 \to \mathcal{S}(0)$ be the dg functor that sends 8 to 4 and 9 to 5; see Definitions 2.17 and 2.19.

Remark 3.11. The proof of Theorem 3.10 decomposes in two steps:

– In the first step we verify the conditions of the recognition Theorem A.3, with $\mathcal{M}$ the category $\mathsf{dgcat}_{\geqslant 0}$, W the class $\mathcal{Q}e_{\geqslant 0}$, J the set of dg functors $\{\zeta(n)\}_{n\geqslant 1}$ and $\underline{k} \to \mathcal{K}$, and I the set of dg functors $\{\iota(n)\}_{n\geqslant 1}$, $\emptyset \to \underline{k}$ and $N : \mathcal{B}_0 \to \mathcal{S}(0)$; see Remark 3.19.
– In the second step, we identify the fibrations in $\mathsf{dgcat}_{\geqslant 0}$ (see Definition 3.8) with the dg functors which have the right lifting property with respect to the elements of the set J; see Proposition 3.20.

We start by observing that the category $\mathsf{dgcat}_{\geqslant 0}$ is complete and cocomplete. Moreover the class $\mathcal{Q}e_{\geqslant 0}$ satisfies the two out of three axioms and it is stable under retracts. We observe also that the domains and codomains of the elements of the sets I and J are small in the category $\mathsf{dgcat}_{\geqslant 0}$. This implies that the first three conditions of the recognition Theorem A.3 are verified.

Lemma 3.12. J-cell $\subseteq \mathcal{Q}e_{\geqslant 0}$.

Proof. Since the inclusion functor

$$\mathsf{dgcat}_{\geqslant 0} \hookrightarrow \mathsf{dgcat}$$

preserves colimits and by Remark 3.7 the class $\mathcal{Q}e_{\geqslant 0}$ consists of those quasi-equivalences in dgcat which belong to $\mathsf{dgcat}_{\geqslant 0}$, the proof follows from [32, Lemme 2.2]. $\square$

We now prove the equality $J\text{-inj} \cap \mathcal{Q}e_{\geqslant 0} = I\text{-inj}$. Consider the following auxiliar class of dg functors.

Definition 3.13. Let $\mathbf{Surj}_{\geqslant 0}$ be the class of dg functors $G : \mathcal{H} \to \mathcal{I}$ in $\mathsf{dgcat}_{\geqslant 0}$ which satisfy the following conditions:

– for all objects $x, y \in \mathcal{H}$, the morphism in $Ch_{\geqslant 0}(k)$

$$G(x, y) : \mathcal{H}(x, y) \longrightarrow \mathcal{I}(Gx, Gy)$$

is a surjective quasi-isomorphism and
– the induced map (see Definition 2.3)

$$G : \mathrm{obj}(\mathcal{H}) \longrightarrow \mathrm{obj}(\mathcal{I})$$

is surjective.

Remark 3.14. In [32, §2], we introduced the class $\mathbf{Surj}$ of dg functors in dgcat. Its definition is the same as Definition 3.13, but we used the base monoidal category $Ch(k)$ instead of $Ch_{\geqslant 0}(k)$. Therefore, the class $\mathbf{Surj}_{\geqslant 0}$ consists of those dg functors in $\mathbf{Surj}$ which belong to $\mathsf{dgcat}_{\geqslant 0}$, under the inclusion functor $\mathsf{dgcat}_{\geqslant 0} \hookrightarrow \mathsf{dgcat}$.

Lemma 3.15. $I\text{-inj} = \mathbf{Surj}_{\geqslant 0}$.

Proof. We show first the inclusion $\supseteq$. Let $G : \mathcal{H} \to \mathcal{I}$ be a dg functor in $\mathbf{Surj}_{\geqslant 0}$. Thanks to Remark 3.14, G belongs to $\mathbf{Surj}$ and so by [32, Lemme 2.3] G has the right lifting property with respect to the dg functors $\{\iota(n)\}_{n\geqslant 1}$ and $\emptyset \to \underline{k}$. Since for all objects $x, y \in \mathcal{H}$, the morphism in $Ch_{\geqslant 0}(k)$

$$G(x, y) : \mathcal{H}(x, y) \longrightarrow \mathcal{I}(Gx, Gy)$$

is surjective in degree zero, the dg functor G has also the right lifting property with respect to the dg functor N. This shows the inclusion $\supseteq$.

We now show the inclusion $\subseteq$. Let $R : \mathcal{C} \to \mathcal{D}$ be a dg functor in I-inj. By [32, Lemme 2.3] R is surjective on objects and:

– for all objects $x, y \in \mathcal{C}$, the morphism in $Ch_{\geqslant 0}(k)$

$$R(x, y) : \mathcal{C}(x, y) \longrightarrow \mathcal{D}(Rx, Ry)$$

is a quasi-isomorphism, which is moreover surjective in positive degrees;
– for all objects $x, y \in \mathcal{C}$, the induced map

$$H_0R(x, y) : H_0\mathcal{C}(x, y) \longrightarrow H_0\mathcal{D}(Rx, Ry)$$

is injective.

Since R belongs to I-inj it has also the right lifting property with respect to the dg functor N. This implies that for all objects $x, y \in \mathcal{C}$, the morphism $R(x, y)$ is also surjective in degree zero. In conclusion, R belongs to $\mathbf{Surj}_{\geqslant 0}$ and so the proof is finished. □

Lemma 3.16. *Let $f : M_\bullet \to N_\bullet$ be a morphism in $Ch_{\geqslant 0}(k)$ such that:*

- *for every $n \geqslant 1$, the map $f_n : M_n \to N_n$ is surjective and*
- *for every $n \geqslant 0$, the induced map $H_n(M_\bullet) \to H_n(N_\bullet)$ is an isomorphism.*

Then the map $f_0 : M_0 \to N_0$ is also surjective.

Proof. Given an $n_0 \in N_0$, we must find an $m_0 \in M_0$ such that $f_0(m_0) = n_0$. By hypothesis, we have an isomorphism $H_0(f) : H_0(M_\bullet) \xrightarrow{\sim} H_0(N_\bullet)$, and so there exist an $m \in M_0$ and an $n_1 \in N_1$ such that $d(n_1) = f_0(m) - n_0$. Since f_1 is surjective, there exists an $m_1 \in M_1$ such that $f_1(m_1) = n_1$. Let $m_0 = m - d(m_1)$. Then $f_0(m_0) = n_0$ as required. □

Lemma 3.17. J-inj $\cap\, \mathcal{Q}e_{\geqslant 0} = \mathbf{Surj}_{\geqslant 0}$.

Proof. The inclusion $\supseteq$ follows from Remark 3.14 and from the inclusion $\supseteq$ in [32, Lemme 2.3]. We now show the inclusion $\subseteq$. Let $R : \mathcal{C} \to \mathcal{D}$ be a dg functor in J-inj $\cap\, \mathcal{Q}e_{\geqslant 0}$. Since R belongs to $\mathcal{Q}e_{\geqslant 0}$ and has the right lifting property with respect to the dg functors $\{\zeta(n)\}_{n \geqslant 1}$, the morphisms in $Ch_{\geqslant 0}(k)$

$$R(x, y) : \mathcal{C}(x, y) \longrightarrow \mathcal{D}(Rx, Ry) \quad x, y \in \mathcal{C}$$

satisfy the conditions of Lemma 3.16. Therefore, these morphisms in $Ch_{\geqslant 0}(k)$ are surjective quasi-isomorphisms. Finally by [32, Lemme 2.3] R is surjective on objects, and so the proof is finished. □

Lemma 3.18. J-cell $\subseteq I$-cof.

Proof. Observe that the dg functors in J-cell have the left lifting property with respect to the class J-inj. Thanks to Lemmas 3.15 and 3.17, we have the equality I-inj $= J$-inj $\cap\, \mathcal{Q}e_{\geqslant 0}$. This implies that the morphisms in J-cell have also the left lifting property with respect to the class I-inj, i.e. J-cell $\subseteq I$-cof. □

Remark 3.19. We have shown that J-cell $\subseteq \mathcal{Q}e_{\geqslant 0} \cap I$-cof (Lemmas 3.12 and 3.18) and that I-inj $= J$-inj $\cap\, \mathcal{Q}_{qe}$ (Lemmas 3.15 and 3.17). This implies that the last three conditions of the recognition Theorem A.3 are satisfied. Therefore the category $\mathsf{dgcat}_{\geqslant 0}$ carries a cofibrantly generated model structure in which $\mathcal{Q}e_{\geqslant 0}$ is the class of weak equivalences, I is a set of cofibrations, and J is a set of generating trivial cofibrations. Recall from Hirschhorn [13, Definition 11.1.2] that the fibrations with respect to this model structure are the dg functors in $\mathsf{dgcat}_{\geqslant 0}$ which have the right lifting property with respect to the elements of the set J. The proof of Theorem 3.10 then follows from the following proposition.

Proposition 3.20. *A dg functor $F : \mathcal{A} \to \mathcal{B}$ in $\mathsf{dgcat}_{\geqslant 0}$ has the right lifting property with respect to the dg functors of the set J if and only if it satisfies conditions* (F1)$_{\geqslant 0}$ *and* (F2)$_{\geqslant 0}$ *of Definition* 3.8.

Proof. The proof is obtained from the proof of Proposition 2.22 by replacing $Ch(k)$ by $Ch_{\geqslant 0}(k)$, condition (F1) by condition (F1)$_{\geqslant 0}$, condition (F2) by condition (F2)$_{\geqslant 0}$, and using condition (F1)$_{\geqslant 0}$ combined with Lemma 3.16 to obtain an induced surjective map

$$\mathcal{C}_{dg}(\mathcal{A})\big(\mathsf{cone}(\hat{u}), \mathsf{cone}(\hat{u})\big) \longrightarrow \mathcal{C}_{\mathsf{dg}}(\mathcal{B})\big(\mathsf{cone}(\hat{v}), \mathsf{cone}(\hat{v})\big)$$

between acyclic dg algebras. □

Lemma 3.21. *Every object in* $\mathsf{dgcat}_{\geqslant 0}$ *is fibrant with respect to the model structure of Theorem* 3.10.

Proof. The proof is obtained from the proof of Lemma 2.24 by replacing condition (F1) by condition (F1)$_{\geqslant 0}$, and condition (F2) by condition (F2)$_{\geqslant 0}$. □

3.2. The truncation functor

Definition 3.22. (Weibel [37, §1.2.7]) Let $M_\bullet$ be a (unbounded) complex of k-modules. Its *intelligent truncation* $\tau_{\geqslant 0}M_\bullet$ is the non-negatively graded complex of k-modules defined as follows:

$$(\tau_{\geqslant 0}M_\bullet)_i = \begin{cases} 0 & \text{if } i < 0, \\ Z_0 & \text{if } i = 0, \\ M_i & \text{if } i > 0 \end{cases}$$

where Z_0 is the kernel of the differential map $d : M_0 \to M_{-1}$.

Remark 3.23. The intelligent truncation construction is functorial in $C_\bullet$ and does not introduce new homology, i.e. we have:

$$H_i(\tau_{\geqslant 0}M_\bullet) = \begin{cases} 0 & \text{if } i < 0, \\ H_i(M_\bullet) & \text{if } i \geqslant 0. \end{cases}$$

This is the reason why we call it intelligent. Moreover, the functor $\tau_{\geqslant 0}$ is lax monoidal (see Section 1.2) and we have an adjunction

$$\begin{tikzcd} Ch(k) \arrow[d, shift left, "\tau_{\geqslant 0}"] \\ Ch_{\geqslant 0}(k). \arrow[u, hook, shift left] \end{tikzcd}$$

Definition 3.24. Let $\mathcal{A}$ be a dg category. The *truncation* $\tau_{\geqslant 0}\mathcal{A}$ of $\mathcal{A}$ is the connective dg category with the same objects as $\mathcal{A}$ and complexes of morphisms given by:

$$(\tau_{\geqslant 0}\mathcal{A})(x, y) := \tau_{\geqslant 0}\mathcal{A}(x, y) \quad x, y \in \mathcal{A}.$$

For x, y and z objects in $\tau_{\geqslant 0}\mathcal{A}$, the composition is given by

$$\tau_{\geqslant 0}\mathcal{A}(y, z) \otimes \tau_{\geqslant 0}\mathcal{A}(x, y) \longrightarrow \tau_{\geqslant 0}\big(\mathcal{A}(y, z) \otimes \mathcal{A}(x, y)\big) \xrightarrow{\tau_{\geqslant 0}(c)} \tau_{\geqslant 0}\mathcal{A}(x, z),$$

where c denotes the composition operation in $\mathcal{A}$. The units in $\tau_{\geqslant 0}\mathcal{A}$ are the same as those of $\mathcal{A}$.

We obtain then an adjunction

$$\begin{tikzcd} \mathsf{dgcat} \arrow[d, shift left, "\tau_{\geqslant 0}"] \\ \mathsf{dgcat}_{\geqslant 0} \arrow[u, hook, shift left, "i"] \end{tikzcd}$$

between dg categories and connective dg categories.

Proposition 3.25. *The preceding adjunction* $(i, \tau_{\geqslant 0})$ *is a Quillen adjunction* (*see Hirschhorn* [13, Definition 8.5.2]), *with respect to the model structures of Theorems* 2.21 *and* 3.10*.*

Proof. By Remark 3.7, the functor i preserves quasi-equivalences. Therefore it remains to show that it preserves also cofibrations. The Quillen model structure of Theorem 3.10 is cofibrantly generated and so by Hirschhorn [13, Proposition 11.2.1], the class of cofibrations equals the class of retracts of its relative I-cells. Since the functor i preserves colimits, it is enough to show that it sends the generating cofibrations in $\mathsf{dgcat}_{\geqslant 0}$ (see Remark 3.11) to cofibrations in dgcat. This is the case for the generating cofibrations $\{\iota(n)\}_{n\geqslant 1}$ and $\emptyset \to \underline{k}$ since they are also generating cofibrations in dgcat; see Notation 2.18. In what concerns the dg functor $i(N) = N$, notice that we can obtain it by the following pushout in dgcat

$$\begin{tikzcd} \mathcal{S}(-1) \arrow[r, "P"] \arrow[d, tail, "\iota(0)"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{B}_0 \arrow[d, "N"] \\ \mathcal{D}(0) \arrow[r] & \mathcal{S}(0), \end{tikzcd}$$

where P sends 4 to 8 and 5 to 9; see Definition 2.17. Since cofibrations are stable under cobase change (see Hirschhorn [13, Proposition 7.2.12]) N is a cofibration in dgcat, and so the proof is finished. $\square$

Proposition 3.26. *Let $\mathcal{A}$ be a cofibrant object in* $\mathsf{dgcat}_{\geqslant 0}$*, with respect to the model structure of Theorem* 3.10*. Then for all objects x and y in $\mathcal{A}$, the non-negatively graded complex $\mathcal{A}(x, y)$ is cofibrant in $Ch_{\geqslant 0}(k)$* (*see Section* 1.1).

Proof. Let $\mathcal{A}$ be a cofibrant object in $\mathsf{dgcat}_{\geqslant 0}$. Thanks to Proposition 3.25 the dg category $i(\mathcal{A}) = \mathcal{A}$ is cofibrant in dgcat, and so by Proposition 2.25 the complexes $\mathcal{A}(x, y)$ are cofibrant in $Ch(k)$. By Lemma 3.16 the trivial fibrations in $Ch_{\geqslant 0}(k)$ are the surjective quasi-isomorphisms. Finally, since we have an inclusion of categories $Ch_{\geqslant 0}(k) \hookrightarrow Ch(k)$, we conclude that the complexes $\mathcal{A}(x, y)$ are also cofibrant in $Ch_{\geqslant 0}(k)$. $\square$

3.3. Path object

Let $\mathcal{B}$ be a dg category. Recall from Definition 2.26 the construction of the path object $P(\mathcal{B})$ for $\mathcal{B}$ in dgcat.

Proposition 3.27. *Let $\mathcal{A}$ be a connective dg category. Then $\tau_{\geqslant 0} P(i(\mathcal{A}))$ is a path object for $\mathcal{A}$* (*see Definition* 2.27)*, with respect to the model structure of Theorem* 3.10.

Proof. Consider the diagonal dg functor

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \times \mathcal{A}$$

in $\mathsf{dgcat}_{\geqslant 0}$. Since the inclusion functor $i : \mathsf{dgcat}_{\geqslant 0} \hookrightarrow \mathsf{dgcat}$ preserves limits, the dg functor $i(\Delta)$ identifies with the diagonal dg functor

$$\Delta : i(\mathcal{A}) \longrightarrow i(\mathcal{A}) \times i(\mathcal{A})$$

in dgcat. Thanks to Proposition 2.28, we have a factorization

$$\begin{tikzcd} i(\mathcal{A}) \arrow[rr, "\Delta"] \arrow[dr, "\gamma"', "\sim"] & & i(\mathcal{A}) \times i(\mathcal{A}) \\ & P(i(\mathcal{A})), \arrow[ur, two heads, "\pi_0 \times \pi_1"'] & \end{tikzcd}$$

with γ a quasi-equivalence and $\pi_0 \times \pi_1$ a fibration in dgcat. The functor

$$\tau_{\geqslant 0} : \mathsf{dgcat} \longrightarrow \mathsf{dgcat}_{\geqslant 0} \quad (\text{see Definition 3.24})$$

preserves quasi-equivalences and by Lemma 3.25 it preserves also fibrations. Since it also preserves limits and the composed functor $\tau_{\geqslant 0} \circ i$ is the identity, we obtain the following factorization

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\Delta"] \arrow[dr, "\tau_{\geqslant 0}(\gamma)"', "\sim"] & & \mathcal{A} \times \mathcal{A} \\ & \tau_{\geqslant 0} P(i(\mathcal{A})) \arrow[ur, two heads, "\tau_{\geqslant 0}(\pi_0) \times \tau_{\geqslant 0}(\pi_1)"'] & \end{tikzcd}$$

in $\mathsf{dgcat}_{\geqslant 0}$. The proof is then finished. $\square$

4. Extended Dold–Kan equivalence

In this section, we will extend the classical Dold–Kan equivalence to a Quillen equivalence between connective dg categories and simplicial k-linear categories; see Theorems 4.16 and 4.30.

Recall from Section 1.1 that the category of simplicial k-modules is denoted by $s\mathbf{Mod}$.

Definition 4.1. A *simplicial k-linear category* is a $s\mathbf{Mod}$-category (see Borceaux [7, Definition 6.2.1]) and a *simplicial k-linear functor* is a $s\mathbf{Mod}$-functor (see [7, Definition 6.2.3]).

Remark 4.2. Note that Definition 4.1 is no more than a combination of Definitions 2.1 and 2.3 in which $Ch(k)$ was replaced with $s\mathbf{Mod}$.

Notation 4.3. We denote by $s\mathbf{Mod}$-Cat the category of simplicial k-linear categories.

4.1. Fixed set of objects

Let I be a (fixed) set.

Notation 4.4. We denote by $Ch_{\geqslant 0}(k)$-Gr_I the category of $Ch_{\geqslant 0}(k)$-graphs with a fixed set of objects I; see Schwede and Shipley [28, §6.2]. An object G in $Ch_{\geqslant 0}(k)$-Gr_I consists of the following data:

– for each pair of objects (x, y) in I, a non-negatively graded complex $G(x, y)$.

A morphism $F : G_1 \to G_2$ in $Ch_{\geqslant 0}(k)$-Gr_I consists of the following data:

– for each pair of objects (x, y) in I, a morphism in $Ch_{\geqslant 0}(k)$
$$F(x, y) : G_1(x, y) \longrightarrow G_2(x, y).$$

Note that the category $Ch_{\geqslant 0}(k)$-Gr_I is simply the product category of copies of $Ch_{\geqslant 0}(k)$ indexed by the set $I \times I$.

Notation 4.5. We denote by $Ch_{\geqslant 0}(k)$-Cat_I the category of $Ch_{\geqslant 0}(k)$-categories with a fixed set of objects I; see Schwede and Shipley [28, §6.1]. An object $\mathcal{A}$ in $Ch_{\geqslant 0}(k)$-Cat_I is a connective dg category (see Definition 3.1) whose set of objects is I. A morphism $F : \mathcal{A} \to \mathcal{B}$ in $Ch_{\geqslant 0}(k)$-Cat_I is a dg functor which is the identity on objects.

We have an adjunction (see Dundas [8, §1.0])

$$\begin{tikzcd} Ch_{\geqslant 0}(k)\text{-}\mathrm{Cat}_I \arrow[d, shift left, "U"] \\ Ch_{\geqslant 0}(k)\text{-}\mathrm{Gr}_I, \arrow[u, shift left, "T_I"] \end{tikzcd} \tag{4.1}$$

where U is the forgetful functor. Its left adjoint T_I is given by

$$T_I(G)(x, y) := \begin{cases} k \oplus \bigoplus_{q \geqslant 0} \bigoplus_{(v_1,\dots,v_q)} G(v_q, y) \otimes \cdots \otimes G(x, v_1) & \text{if } x = y, \\ \bigoplus_{q \geqslant 0} \bigoplus_{(v_1,\dots,v_q)} G(v_q, y) \otimes \cdots \otimes G(x, v_1) & \text{if } x \neq y \end{cases}$$

where $(v_1, \dots, v_q)$ is a ordered sequence of objects in I. Composition is given by concatenation of terms, with $k \subset T_I(G)(x, x)$ acting as the unit via the canonical isomorphism $k \otimes G(x, y) \xrightarrow{\sim} G(x, y)$. The unit $G \to UT_I(G)$ of the adjunction is given by the inclusion of morphisms and the counit $T_I U(\mathcal{A}) \to \mathcal{A}$ of the adjunction is given by the composition of morphisms. Given objects x, y and z in I, the composition morphism in $Ch_{\geqslant 0}(k)$

$$T_I(G)(y, z) \otimes T_I(G)(x, y) \longrightarrow T_I(G)(x, z) \quad \text{(see Definition 2.1)} \tag{4.2}$$

can be described explicitly as follows: given an integer $q \geqslant 0$ and an ordered sequence $(v_1, \dots, v_q)$ of objects in I, we have the non-negatively graded complex

$$G(v_q, y) \otimes \cdots \otimes G(x, v_1). \tag{4.3}$$

Given another integer $q' \geqslant 0$ and an ordered sequence $(w_1, \dots, w_{q'})$ of objects in I, we have the non-negatively graded complex

$$G(w_{q'}, z) \otimes \cdots \otimes G(y, w_1). \tag{4.4}$$

The restriction of the composition morphism (4.2) to the non-negatively graded complexes (4.3) with (4.4) is given by the canonical isomorphism

$$\begin{tikzcd} (G(w_{q'}, z) \otimes \cdots \otimes G(y, w_1)) \otimes (G(v_q, y) \otimes \cdots \otimes G(x, v_1)) \arrow[d, "\sim"] \\ G(w_{q'}, z) \otimes \cdots \otimes G(y, w_1) \otimes G(v_q, y) \otimes \cdots \otimes G(x, v_1) \end{tikzcd}$$

followed by the inclusion of morphisms. When $x = y$ and the non-negatively graded complex (4.3) is $k \subset T_I(G)(x, x)$, we use the canonical isomorphism

$$G(w_{q'}, z) \otimes \cdots \otimes G(x, w_1) \otimes k \xrightarrow{\sim} G(w_{q'}, z) \otimes \cdots \otimes G(x, w_1).$$

Similarly, when $y = z$ and the non-negatively graded complex (4.4) is $k \subset T_I(G)(y, y)$, we use the canonical isomorphism

$$k \otimes G(v_q, y) \otimes \cdots \otimes G(x, v_1) \xrightarrow{\sim} G(v_q, y) \otimes \cdots \otimes G(x, v_1).$$

Remark 4.6. The category $Ch_{\geqslant 0}(k)$-Gr_I admits a standard Quillen model structure (see Schwede and Shipley [28, §6.2]):

- the weak equivalences are the morphisms $F : G_1 \to G_2$ such that for each pair of objects $(x, y) \in I$, the morphism in $Ch_{\geqslant 0}(k)$

  $$F(x, y) : G_1(x, y) \longrightarrow G_2(x, y)$$

  is a quasi-isomorphism;
- the fibrations are the morphisms $F : G_1 \to G_2$ such that for each pair of objects $(x, y) \in I$, the morphism in $Ch_{\geqslant 0}(k)$

  $$F(x, y) : G_1(x, y) \longrightarrow G_2(x, y)$$

  is surjective in positive degrees.

The category $Ch_{\geqslant 0}(k)$-Cat_I admits also a standard Quillen model structure; see Schwede and Shipley [28, Proposition 6.3]. Its weak equivalences are the dg functors $F : \mathcal{A} \to \mathcal{B}$ such that $U(F)$ is a weak equivalence in $Ch_{\geqslant 0}(k)$-Gr_I. Similarly, its fibrations are the dg functors F such that $U(F)$ is a fibration in $Ch_{\geqslant 0}(k)$-Gr_I.

Remark 4.7. By replacing in Notation 4.4 the monoidal category $Ch_{\geqslant 0}(k)$ by $s\mathbf{Mod}$, we obtain the category $s\mathbf{Mod}$-Gr_I of $s\mathbf{Mod}$-graphs with a fixed set of objects I. Similarly, by replacing in Notation 4.5 the monoidal category $Ch_{\geqslant 0}(k)$ by $s\mathbf{Mod}$, we obtain the category $s\mathbf{Mod}$-Cat_I of simplicial k-linear categories with a fixed set of objects I. Furthermore, by taking in the preceding adjunction (4.1) the levelwise tensor product of simplicial k-modules $\wedge$ instead of the tensor product of complexes, we obtain an adjunction

$$\begin{tikzcd} s\mathbf{Mod}\text{-}\mathrm{Cat}_I \arrow[d, shift left, "U"] \\ s\mathbf{Mod}\text{-}\mathrm{Gr}_I . \arrow[u, shift left, "T_I"] \end{tikzcd} \tag{4.5}$$

As in Remark 4.6 the categories $s\mathbf{Mod}$-Gr_I and $s\mathbf{Mod}$-Cat_I admit a standard Quillen model structure; see Schwede and Shipley [28, §6.2].

- The weak equivalences in $s\mathbf{Mod}$-Gr_I are the morphisms $F : A \to B$ such that for each pair of objects $(x, y) \in I$, the morphism in $s\mathbf{Mod}$

  $$F(x, y) : A(x, y) \longrightarrow B(x, y)$$

  is a weak equivalence (see Section 1.1).
- The fibrations in $s\mathbf{Mod}$-Gr_I are the morphisms $F : A \to B$ such that for each pair of objects $(x, y) \in I$, the morphism in $s\mathbf{Mod}$

  $$F(x, y) : A(x, y) \longrightarrow B(x, y)$$

  is a fibration (see Section 1.1).

A weak equivalence in $s\mathbf{Mod}$-Cat_I is a simplicial k-linear functor $F : A \to B$ such that $U(F)$ is a weak equivalence in $s\mathbf{Mod}$-Gr_I. Similarly, a fibration in $s\mathbf{Mod}$-Cat_I is a simplicial k-linear functor F such that $U(F)$ is a fibration in $s\mathbf{Mod}$-Gr_I.

4.2. Dold–Kan equivalence

Recall from Goerss and Jardine [15, §III-2.3] the Dold–Kan equivalence between simplicial k-modules and non-negatively graded complexes

$$\begin{tikzcd} s\mathbf{Mod} \arrow[d, shift left, "N"] \\ Ch_{\geqslant 0}(k), \arrow[u, shift left, "\Gamma"] \end{tikzcd} \tag{4.6}$$

where N is the normalization functor and Γ its inverse. Given a set I, the Dold–Kan equivalence induces an equivalence of categories (still denote by N and Γ)

$$\begin{tikzcd} s\mathbf{Mod}\text{-}\mathrm{Gr}_I \arrow[d, shift left, "N"] \\ Ch_{\geqslant 0}(k)\text{-}\mathrm{Gr}_I . \arrow[u, shift left, "\Gamma"] \end{tikzcd} \tag{4.7}$$

The normalization functor $N : s\mathbf{Mod} \to Ch_{\geqslant 0}(k)$ is lax monoidal (see Section 1.2) via the Eilenberg–Mac Lane shuffle map (see Mac Lane [25, §VIII-8.8]):

$$\nabla : NA \otimes NB \longrightarrow N(A \wedge B) \quad A, B \in s\mathbf{Mod},$$

and so it induces a normalization functor (see Notations 4.5 and 4.7)

$$\begin{array}{c} s\mathbf{Mod}\text{-}\mathrm{Cat}_I \\ \Big\downarrow{\scriptstyle N_I} \\ Ch_{\geqslant 0}(k)\text{-}\mathrm{Cat}_I. \end{array}$$

Given $A \in s\mathbf{Mod}\text{-}\mathrm{Cat}_I$ and x, y and z objects in A, $N_I(A)$ has the same objects as A and the complexes of morphisms

$$N_I(A)(x, y) := N\big(A(x, y)\big) \quad x, y \in A.$$

Composition is given by

$$N\big(A(y,z)\big) \otimes N\big(A(x,y)\big) \xrightarrow{\nabla} N\big(A(y,z) \wedge A(x,y)\big) \xrightarrow{N(c)} NA(x,z),$$

where c denotes the composition operation in A. The units in $N_I(A)$ are induced by those of A under the normalization functor N.

Thanks to Schwede and Shipley [28, §4.2] the lax monoidal structure on the normalization functor $N : s\mathbf{Mod} \to Ch_{\geqslant 0}(k)$, given by the Eilenberg–Mac Lane shuffle map ∇, induces a comonoidal structure on its left adjoint functor (see Section 1.2):

$$\tilde{\nabla} : \Gamma(M_\bullet \otimes N_\bullet) \longrightarrow \Gamma(M_\bullet) \wedge \Gamma(N_\bullet) \quad M_\bullet, N_\bullet \in Ch_{\geqslant 0}(k).$$

As shown in [28, §3.3], this comonoidal structure allows us to construct the left adjoint functor L_I to N_I as follows: the value of L_I at $\mathcal{A} \in Ch_{\geqslant 0}(k)\text{-}\mathrm{Cat}_I$ is given by the coequalizer of two morphisms in $s\mathbf{Mod}\text{-}\mathrm{Cat}_I$

$$T_I \Gamma U T_I U(\mathcal{A}) \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} T_I \Gamma U(\mathcal{A}) \longrightarrow L_I(\mathcal{A}).$$

The morphism ψ_1 is the unique morphism in $s\mathbf{Mod}\text{-}\mathrm{Cat}_I$ induced by the morphism

$$\Gamma U T_I U(\mathcal{A}) \longrightarrow U T_I \Gamma U(\mathcal{A})$$

in $s\mathbf{Mod}\text{-}\mathrm{Gr}_I$, whose value at $\Gamma U T_I U(\mathcal{A})(x, y)$ is given by

$$\begin{array}{c} \displaystyle\bigoplus_{q \geqslant 0} \bigoplus_{(v_1,\ldots,v_q)} \Gamma(\mathcal{A}(v_q, y) \otimes \cdots \otimes \mathcal{A}(x, v_1)) \\ \Big\downarrow{\scriptstyle \bigoplus_{q\geqslant 0} \bigoplus_{(v_1,\ldots,v_q)} \tilde{\nabla}} \\ \displaystyle\bigoplus_{q \geqslant 0} \bigoplus_{(v_1,\ldots,v_q)} \Gamma\mathcal{A}(v_q, y) \wedge \cdots \wedge \Gamma\mathcal{A}(x, v_1). \end{array}$$

The morphism ψ_2 is obtained from the counit of the adjunction

$$T_I U(\mathcal{A}) \longrightarrow \mathcal{A}$$

by applying the composite functor $T_I \Gamma U$. In sum, given a set I, we have an adjunction

$$\begin{array}{c} s\mathbf{Mod}\text{-}\mathrm{Cat}_I \\ {\scriptstyle L_I}\Big\uparrow \Big\downarrow{\scriptstyle N_I} \\ Ch_{\geqslant 0}(k)\text{-}\mathrm{Cat}_I. \end{array} \tag{4.8}$$

4.3. Left adjoint

The normalization functor

$$N_I : s\mathbf{Mod}\text{-}\mathrm{Cat}_I \longrightarrow Ch_{\geqslant 0}(k)\text{-}\mathrm{Cat}_I$$

is defined for every set I, and so it gives rise to a 'global' normalization functor

$$N : s\mathbf{Mod}\text{-}\mathrm{Cat} \longrightarrow \mathsf{dgcat}_{\geqslant 0}. \tag{4.9}$$

In this subsection we will construct its left adjoint functor

$$L : \mathsf{dgcat}_{\geqslant 0} \longrightarrow s\mathbf{Mod}\text{-}\mathrm{Cat}.$$

We start by defining two auxiliar functors.

Definition 4.8 *(Auxiliar functor Θ).* Let $\mathcal{A}$ and $\mathcal{A}'$ be two connective dg categories. We denote by I the set of objects of $\mathcal{A}$ and by I' the set of objects of $\mathcal{A}'$. We define $\Theta(\mathcal{A})$ as the simplicial k-linear category $T_I \Gamma U(\mathcal{A})$ (belonging to $s\mathbf{Mod}\text{-Cat}_I$) and $\Theta(\mathcal{A}')$ as the simplicial k-linear category $T_{I'} \Gamma U(\mathcal{A}')$ (belonging to $s\mathbf{Mod}\text{-Cat}_{I'}$); see the adjunctions (4.1), (4.5) and (4.7).

Let $F : \mathcal{A} \to \mathcal{A}'$ be a dg functor in $\mathsf{dgcat}_{\geqslant 0}$. Since we have the equalities

$$\mathrm{obj}\big(T_I \Gamma U(\mathcal{A})\big) = \mathrm{obj}(\mathcal{A}) \qquad \mathrm{obj}\big(T_{I'} \Gamma U\big(\mathcal{A}'\big)\big) = \mathrm{obj}\big(\mathcal{A}'\big)$$

we define the map

$$\mathrm{obj}\big(T_I \Gamma U(\mathcal{A})\big) \longrightarrow \mathrm{obj}\big(T_{I'} \Gamma U\big(\mathcal{A}'\big)\big)$$

to be the same one as F. Recall that for all objects $x, y \in T_I \Gamma U(\mathcal{A})$, we have

$$T_I \Gamma U(\mathcal{A})(x, y) = \bigoplus_{q \geqslant 0} \bigoplus_{(v_1, \dots, v_q)} \Gamma \mathcal{A}(v_q, y) \wedge \cdots \wedge \Gamma \mathcal{A}(x, v_1)$$

and

$$T_{I'} \Gamma U\big(\mathcal{A}'\big)(Fx, Fy) = \bigoplus_{q' \geqslant 0} \bigoplus_{(w_1, \dots, w_{q'})} \Gamma \mathcal{A}'(w_{q'}, Fy) \wedge \cdots \wedge \Gamma \mathcal{A}'(Fx, w_1).$$

We define the morphism in $Ch_{\geqslant 0}(k)$

$$T_I \Gamma U(\mathcal{A})(x, y) \longrightarrow T_{I'} \Gamma U\big(\mathcal{A}'\big)(Fx, Fy)$$

to be the one whose value at each component is given by

$$\begin{array}{c} \Gamma \mathcal{A}(v_q, y) \wedge \cdots \wedge \Gamma \mathcal{A}(x, v_1) \\ \Big\downarrow {\scriptstyle \Gamma F(v_q, y) \wedge \cdots \wedge \Gamma F(x, v_1)} \\ \Gamma \mathcal{A}'(F v_q, F y) \wedge \cdots \wedge \Gamma \mathcal{A}'(Fx, F v_1). \end{array}$$

In sum, we obtain a well-defined functor

$$\Theta : \mathsf{dgcat}_{\geqslant 0} \longrightarrow s\mathbf{Mod}\text{-Cat} \qquad \mathcal{A} \longmapsto T_I \Gamma U(\mathcal{A}).$$

Definition 4.9 *(Auxiliar functor $\mathbb{H}$).* Let $\mathcal{A}$ and $\mathcal{A}'$ be two connective dg categories. We denote by I the set of objects of $\mathcal{A}$ and by I' the set of objects of $\mathcal{A}'$. We define $\mathbb{H}(\mathcal{A})$ as the connective dg category $T_I U(\mathcal{A})$ (belonging to $Ch_{\geqslant 0}(k)\text{-Cat}_I$) and $\mathbb{H}(\mathcal{A}')$ as the connective dg category $T_{I'} U(\mathcal{A}')$ (belonging to $Ch_{\geqslant 0}(k)\text{-Cat}_{I'}$); see the adjunction (4.1).

Let $F : \mathcal{A} \to \mathcal{A}'$ be a dg functor in $\mathsf{dgcat}_{\geqslant 0}$. Since we have the equalities

$$\mathrm{obj}\big(T_I U(\mathcal{A})\big) = \mathrm{obj}(\mathcal{A}) \qquad \mathrm{obj}\big(T_{I'} U\big(\mathcal{A}'\big)\big) = \mathrm{obj}\big(\mathcal{A}'\big)$$

we define the map

$$\mathrm{obj}\big(T_I U(\mathcal{A})\big) \longrightarrow \mathrm{obj}\big(T_{I'} U\big(\mathcal{A}'\big)\big)$$

to be the same one as F. Recall that for all objects $x, y \in T_I U(\mathcal{A})$, we have

$$T_I U(\mathcal{A})(x, y) = \bigoplus_{q \geqslant 0} \bigoplus_{(v_1, \dots, v_q)} \mathcal{A}(v_q, y) \otimes \cdots \otimes \mathcal{A}(x, v_1)$$

and

$$T_{I'} U\big(\mathcal{A}'\big)(Fx, Fy) = \bigoplus_{q' \geqslant 0} \bigoplus_{(w_1, \dots, w_{q'})} \mathcal{A}'(w_{q'}, Fy) \otimes \cdots \otimes \mathcal{A}'(Fx, w_1).$$

We define the morphism in $Ch_{\geqslant 0}(k)$

$$T_I U(\mathcal{A})(x, y) \longrightarrow T_{I'} U\big(\mathcal{A}'\big)(Fx, Fy)$$

to be the one whose value at each component is given by

$$\begin{array}{c} \mathcal{A}(v_q, y) \otimes \cdots \otimes \mathcal{A}(x, v_1) \\ \Big\downarrow {\scriptstyle F(v_q, y) \otimes \cdots \otimes F(x, v_1)} \\ \mathcal{A}'(F v_q, F y) \otimes \cdots \otimes \mathcal{A}'(Fx, F v_1). \end{array}$$

In sum, we obtain a well-defined functor

$$\mathbb{H} : \mathsf{dgcat}_{\geqslant 0} \longrightarrow \mathsf{dgcat}_{\geqslant 0} \qquad \mathcal{A} \longmapsto T_I U(\mathcal{A}).$$

Now, let $\mathcal{A}$ and $\mathcal{A}'$ be two connective dg categories. We denote by I the set of objects of $\mathcal{A}$ and by I' the set of objects of $\mathcal{A}'$. We define $L(\mathcal{A})$ as the simplicial k-linear category $L_I(\mathcal{A})$. Similarly, we define $L(\mathcal{A}')$ as the simplicial k-linear category $L_{I'}(\mathcal{A}')$; see adjunction (4.8). Given a dg functor $F : \mathcal{A} \to \mathcal{A}'$ in $\mathsf{dgcat}_{\geqslant 0}$, it gives rise to simplicial k-linear functors (see Definitions 4.8 and 4.9)

$$\Theta(F) : T_I \Gamma U(\mathcal{A}) \longrightarrow T_{I'} \Gamma U\big(\mathcal{A}'\big)$$

and

$$\Theta\big(\mathbb{H}(F)\big) : T_I \Gamma U T_I U(\mathcal{A}) \longrightarrow T_{I'} \Gamma U T_{I'} U\big(\mathcal{A}'\big).$$

We obtain then a (solid) diagram in $s\mathbf{Mod}$-Cat:

$$\begin{tikzcd}
T_I \Gamma U T_I U(\mathcal{A}) \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] \arrow[d, "\Theta(\mathbb{H}(F))"'] & T_I \Gamma U(\mathcal{A}) \arrow[r] \arrow[d, "\Theta(F)"] & L_I(\mathcal{A}) =: L(\mathcal{A}) \arrow[d, dotted, "L(F)"] \\
T_{I'} \Gamma U T_{I'} U(\mathcal{A}') \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] & T_{I'} \Gamma U(\mathcal{A}') \arrow[r] & L_{I'}(\mathcal{A}') =: L(\mathcal{A}'),
\end{tikzcd}$$

whose squares involving ψ_1 and ψ_2 are commutative. Since the functors

$$s\mathbf{Mod}\text{-}\mathrm{Cat}_I \longrightarrow s\mathbf{Mod}\text{-}\mathrm{Cat} \quad \text{and} \quad s\mathbf{Mod}\text{-}\mathrm{Cat}_{I'} \longrightarrow s\mathbf{Mod}\text{-}\mathrm{Cat}$$

preserve coequalizers, the previous diagram in $s\mathbf{Mod}$-Cat induces a simplicial k-linear functor

$$L(F) : L(\mathcal{A}) \longrightarrow L\big(\mathcal{A}'\big).$$

In sum, we obtain a well-defined functor

$$L : \mathsf{dgcat}_{\geqslant 0} \longrightarrow s\mathbf{Mod}\text{-}\mathrm{Cat} \qquad \mathcal{A} \longmapsto L(\mathcal{A}).$$

Proposition 4.10. *The functor L is left adjoint to the normalization functor*

$$N : s\mathbf{Mod}\text{-}\mathrm{Cat} \longrightarrow \mathsf{dgcat}_{\geqslant 0}.$$

Proof. Let $\mathcal{A}$ be a connective dg category and B a simplicial k-linear category. We denote by I the set of objects of $\mathcal{A}$. Using the adjunction (4.8)

$$\begin{tikzcd}
s\mathbf{Mod}\text{-}\mathrm{Cat}_I \arrow[d, shift left, "N_I"] \\
Ch_{\geqslant 0}(k)\text{-}\mathrm{Cat}_I, \arrow[u, shift left, "L_I"]
\end{tikzcd}$$

we will construct two natural maps

$$\mathsf{Hom}_{s\mathbf{Mod}\text{-}\mathrm{Cat}}(L(\mathcal{A}), B) \underset{\eta}{\overset{\phi}{\rightleftarrows}} \mathsf{Hom}_{\mathsf{dgcat}_{\geqslant 0}}(\mathcal{A}, N(B))$$

and show that they are inverse of each other.

We start by describing the map ϕ. Let $G : L(\mathcal{A}) \to B$ be a simplicial k-linear functor. Consider the simplicial k-linear category B_G with objects

$$\mathrm{obj}(B_G) = \big\{(a, Ga) \;\big|\; a \in L(\mathcal{A})\big\}$$

and simplicial k-modules of morphisms given by

$$B_G\big((a, Ga), \big(a', Ga'\big)\big) := B\big(Ga, Ga'\big).$$

The composition is induced by the one on B. We have a factorization

$$\begin{tikzcd}
L(\mathcal{A}) \arrow[rr, "G"] \arrow[dr, "\widetilde{G}"'] & & B \\
& B_G, \arrow[ur, "G_{obj}"'] &
\end{tikzcd}$$

where $\widetilde{G}$ maps a to (a, Ga) and G_{obj} maps (a, Ga) to Ga. The simplicial k-linear functor $\widetilde{G}$ induces the identity map on objects and so it belongs to $s\mathbf{Mod}\text{-}\mathrm{Cat}_I$. Note that the construction of B_G (and of G_{obj}) depends only on the map of sets

$$G : \mathrm{obj}\big(L(\mathcal{A})\big) \longrightarrow \mathrm{obj}(B).$$

Finally, the dg functor $\phi(G)$ in $\mathsf{dgcat}_{\geqslant 0}$ is given by the following composition

$$\mathcal{A} \xrightarrow{\widetilde{G}^{\natural}} N(B_G) \xrightarrow{N(G_{obj})} N(B),$$

where $\widetilde{G}^{\natural}$ is the dg functor corresponding to $\widetilde{G}$ under the adjunction (L_I, N_I).

We now describe the map η. Recall that the normalization functor N does not alter the set of objects. Therefore, given a dg functor $F : \mathcal{A} \to N(B)$ in $\mathsf{dgcat}_{\geqslant 0}$ we can consider the simplicial k-linear category B_F with set of objects

$$\mathrm{obj}(B_F) = \big\{(a, Fa) \,\big|\, a \in \mathcal{A}\big\}$$

and simplicial k-modules of morphisms given by

$$B_F\big((a, Fa), \big(a', Fa'\big)\big) := B\big(Fa, Fa'\big).$$

The composition is induced by the one on B. We have a factorization

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "\widetilde{F}"'] & & N(B) & & & B \\ & N(B_F) \arrow[ur, "N(F_{obj})"'] & & & B_F, \arrow[ur, "F_{obj}"'] & \end{tikzcd}$$

where $\widetilde{F}$ maps a to (a, Fa) and F_{obj} maps (a, Fa) to $F(a)$. The dg functor $\widetilde{F}$ induces the identity map on objects and so it belongs to $Ch_{\geqslant 0}(k)\text{-}\mathrm{Cat}_I$. Note that, as before, the construction of B_F (and of F_{obj}) only depends on the map of sets

$$F : \mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}\big(N(B)\big).$$

Finally, the simplicial k-linear functor $\eta(F)$ is given by the following composition

$$L(\mathcal{A}) \xrightarrow{\widetilde{F}^{\sharp}} B_F \xrightarrow{F_{obj}} B,$$

where $\widetilde{F}^{\natural}$ is the simplicial k-linear functor corresponding to $\widetilde{F}$ under the adjunction (L_I, N_I).

We now show that the maps ϕ and η are inverse of each other. Given a dg functor $F : \mathcal{A} \to N(B)$ in $\mathsf{dgcat}_{\geqslant 0}$, consider the simplicial k-linear functor

$$\eta(F) : L(\mathcal{A}) \xrightarrow{\widetilde{F}^{\sharp}} B_F \xrightarrow{F_{obj}} B.$$

Since F and $\eta(F)$ induce the same map on objects, B_F identifies with $B_{\eta(F)}$, F_{obj} identifies with $\eta(F)_{obj}$, and $\widetilde{F}^{\sharp}$ identifies with $\widetilde{\eta(F)}$. Using the adjunction (L_I, N_I), we conclude that F identifies with $(\phi \circ \eta)(F)$ and so we have the equality $\phi \circ \eta = \mathrm{Id}$. Conversely, given a simplicial k-linear functor $G : L(\mathcal{A}) \to B$, consider the dg functor

$$\phi(G) : \mathcal{A} \xrightarrow{\widetilde{G}^{\natural}} N(B_G) \xrightarrow{N(G_{obj})} N(B).$$

Since G and $\phi(G)$ induce the same map on objects, B_G identifies with $B_{\phi(G)}$, G_{obj} identifies with $\phi(G)_{obj}$, and $\widetilde{G}^{\natural}$ identifies with $\widetilde{\phi(G)}$. Using once again the adjunction (L_I, N_I), we conclude that F identifies with $(\eta \circ \phi)(G)$ and so $\eta \circ \phi = \mathrm{Id}$. The proof is then finished. $\square$

4.4. Quillen model structure

In this subsection we will lift (see Theorem 4.16) the Quillen model structure on $\mathsf{dgcat}_{\geqslant 0}$ of Theorem 3.10 along the adjunction (see Proposition 4.10)

$$\begin{tikzcd} s\mathbf{Mod}\text{-}\mathrm{Cat} \arrow[d, shift left, "N"] \\ \mathsf{dgcat}_{\geqslant 0}. \arrow[u, shift left, "L"] \end{tikzcd}$$

Definition 4.11. Let A be a simplicial k-linear category (see Definition 4.1). The *category* $\pi_0(A)$ has the same objects as A and morphisms given by $\pi_0(A)(x, y) := \pi_0(A(x, y))$, where $\pi_0(A(x, y))$ is the set of connected components of the simplicial k-linear set $A(x, y)$.

Remark 4.12. For all objects $x, y \in A$, we have an exact sequence of k-modules

$$A(x, y)_1 \xrightarrow{d_0-d_1} A(x, y)_0 \longrightarrow \pi_0\big(A(x, y)\big) \longrightarrow 0.$$

Definition 4.13. A simplicial k-linear functor $F : A \to B$ is a *weak equivalence* if:

(WE1) for all objects $x, y \in A$, the morphism in $s\mathbf{Mod}$

$$F(x, y) : A(x, y) \longrightarrow B(Fx, Fy)$$

is a weak equivalence (see Section 1.1) and

(WE2) the induced functor (see Definition 4.11)

$$\pi_0(F) : \pi_0(A) \longrightarrow \pi_0(B)$$

is an equivalence of categories.

Definition 4.14. A simplicial k-linear functor $F : A \to B$ is a *fibration* if:

(k-F1) for all objects $x, y \in \mathcal{A}$, the morphism in $s\mathbf{Mod}$

$$F(x, y) : A(x, y) \longrightarrow B(Fx, Fy)$$

is a fibration (see Section 1.1) and

(k-F2) given an object x in A and a morphism $v : Fx \to y$ in B, which becomes invertible in $\pi_0(B)$, there exists a morphism $u : x \to x'$ in A, which becomes invertible in $\pi_0(A)$, such that $Fu = v$.

Definition 4.15. A simplicial k-linear functor $F : A \to B$ is a *cofibration* if it has the left lifting property with respect to the simplicial k-linear functors which are simultaneously fibrations and weak equivalences.

Theorem 4.16. *The category* $s\mathbf{Mod}$-Cat *endowed with the notions of weak equivalence, fibration and cofibration of Definitions* 4.13, 4.14*, and* 4.15 *is a* (*cofibrantly generated*) *Quillen model category. Moreover, the adjunction* (L, N) *is a Quillen adjunction.*

Proof of Theorem 4.16. The proof of Theorem 4.16 decomposes in two steps:

– In the first step we show that the weak equivalences and fibrations of Definitions 4.13 and 4.14 coincide, under the normalization functor $N : s\mathbf{Mod}\text{-Cat} \to \mathsf{dgcat}_{\geqslant 0}$, with the quasi-equivalences and fibrations of Definitions 3.5 and 3.8; see Propositions 4.18 and 4.19.
– In the second step, we verify the conditions of Theorem A.2, with respect to the adjunction (L, N) and the Quillen model structure of Theorem 3.10; see Proposition 4.27.

Lemma 4.17. *Let A be a simplicial k-linear category, x and y objects in A, and $f : x \to y$ a morphism in A. Then, f is invertible in $\pi_0(A)$ if and only if Nf is a homotopy equivalence* (*see Definition* 2.6) *in NA.*

Proof. We start by observing that if we restrict ourselves to the 0-simplex morphisms in A and to the degree zero morphisms in NA, we obtain the same category. In fact, the degree zero component of the shuffle map ∇ (see Mac Lane [25, §VIII-8.8]) used in the definition of NA is the identity map.

Suppose that f is invertible in $\pi_0(A)$. Then there exist a morphism $g : y \to x$ and morphisms $h_x \in A(x, x)_1$ and $h_y \in A(y, y)_1$, such that $d_0(h_x) = \mathbf{1}_x$, $d_1(h_x) = g \circ f$, $d_0(h_y) = \mathbf{1}_y$ and $d_1(h_y) = f \circ g$. Note that the image of h_x under the normalization functor N is a degree one morphism in $NA(x, x)$, whose differential is $g \circ f - \mathbf{1}_x$. Similarly, the image of h_y under the normalization functor N is a degree one morphism in $NA(y, y)$, whose differential is $f \circ g - \mathbf{1}_y$. This implies that Nf is invertible in $\mathsf{H}_0(NA)$, and so a homotopy equivalence in NA.

Now, suppose that Nf is a homotopy equivalence in NA. Then there exist a morphism $g : y \to x$ and morphisms $h_x \in NA(x, x)_1$ and $h_y \in NA(y, y)_1$, such that $d(h_x) = g \circ f - \mathbf{1}_x$ and $d(h_y) = f \circ g - \mathbf{1}_y$. Note that g corresponds to a morphism in A and h_x and h_y correspond to 1-simplex morphisms in A. Since $d(h_x) = (d_0 - d_1)(h_x)$ and $d(h_y) = (d_0 - d_1)(h_x)$, we conclude by Remark 4.12 that the morphism $g \circ f$ is the identity of x in $\pi_0(A)$ and the morphism $f \circ g$ is the identity of y in $\pi_0(B)$. Therefore f is invertible in $\pi_0(A)$, and so the proof is finished. □

Proposition 4.18. *A simplicial k-linear functor $F : A \to B$ is a weak equivalence* (*see Definition* 4.13) *if and only if the dg functor $NF : NA \to NB$ in* $\mathsf{dgcat}_{\geqslant 0}$ *is a quasi-equivalence* (*see Definition* 3.5).

Proof. We show that condition (WE1) is equivalent to condition $(\mathrm{QE1})_{\geqslant 0}$ and that condition (WE2) is equivalent to condition $(\mathrm{QE2})_{\geqslant 0}$. For all objects $x, y \in A$ and $i \geqslant 0$, we have by the Dold–Kan equivalence [15, §III-2.3], a commutative diagram

$$\begin{tikzcd} \pi_i A(x,y) \arrow[r, "F"] \arrow[d, "\sim"'] & \pi_i B(Fx, Fy) \arrow[d, "\sim"] \\ H_i(NA(x,y)) \arrow[r, "NF"'] & H_i(NB(Fx, Fy)), \end{tikzcd}$$

where the vertical arrows are isomorphisms. This shows that condition (WE1) is equivalent to condition $(\mathrm{QE1})_{\geqslant 0}$.

Suppose that $\pi_0(F)$ is essentially surjective. We need to show that the induced functor

$$\mathsf{H}_0(NF) : \mathsf{H}_0(NA) \longrightarrow \mathsf{H}_0(NB)$$

is also essentially surjective. Let z be an object in $\mathsf{H}_0(NB)$. Since $\pi_0(B)$ and $\mathsf{H}_0(NB)$ have the same objects, we can consider z as an object in $\pi_0(B)$. By hypothesis, $\pi_0(F)$ is essentially surjective and so there exists an object w in A and a morphism

$$f : Fw \to z$$

in B, which becomes invertible in $\pi_0(B)$. Since A and NA have the same objects, we can consider w as an object in NA and the morphism $Nf : NFw \to z$ in NB. Thanks to Lemma 4.17, Nf becomes invertible in $\mathsf{H}_0(NB)$ and so we conclude that the functor $\mathsf{H}_0(NF)$ is essentially surjective.

Now, suppose that $\mathsf{H}_0(NF)$ is essentially surjective. We need to show that the induced functor

$$\pi_0(F) : \pi_0(A) \longrightarrow \pi_0(B)$$

is also essentially surjective. Let z be an object in $\pi_0(B)$. The categories $\pi_0(B)$ and $\mathsf{H}_0(NB)$ have the same objects, and so we can consider z as an object in $\mathsf{H}_0(NB)$. Since $\mathsf{H}_0(NF)$ is essentially surjective, there exists an object w in NA and a morphism

$$f : NFw \to z$$

in NB, which becomes invertible in $\mathsf{H}_0(NB)$. When we restrict ourselves to the 0-simplex morphisms in A and to the degree zero morphisms in NA we obtain the same category, and similarly for B and NB. In fact, the degree zero component of the shuffle map ∇ (see Mac Lane [25, §VIII-8.8]) is the identity map. Therefore, we can consider w as an object in A and f as a morphism $Fw \to z$ in B. By Lemma 4.17, f becomes invertible in $\pi_0(B)$ and so $\pi_0(G)$ is essentially surjective. This shows that condition (WE2) is equivalent to condition $(\mathrm{QE2})_{\geqslant 0}$, and so the proof is finished. □

Proposition 4.19. *A simplicial k-linear functor $F : A \to B$ is a fibration (see Definition 4.14) if and only if the dg functor $NF : NA \to NB$ in $\mathsf{dgcat}_{\geqslant 0}$ is a fibration (see Definition 3.8).*

Proof. We show that condition (k-F1) is equivalent to condition $(\mathrm{F}_1)_{\geqslant 0}$ and that condition (k-F2) is equivalent to condition $(\mathrm{F2})_{\geqslant 0}$. Since the Dold–Kan equivalence [15, §III-2.3]

$$\begin{tikzcd} s\mathbf{Mod} \arrow[d, shift left, "N"] \\ Ch_{\geqslant 0}(k) \arrow[u, shift left, "\Gamma"] \end{tikzcd}$$

is a Quillen adjunction, the normalization functor N preserves fibrations. Therefore condition (k-F1) implies condition $(\mathrm{F1})_{\geqslant 0}$. Conversely, Goerss and Jardine [15, Lemma 2.11(2)-III] show that condition $(\mathrm{F1})_{\geqslant 0}$ implies condition (k-F1).

Suppose that F satisfies condition (k-F2). Let x be an object in NA and

$$v : NFx \longrightarrow y$$

a morphism in NB which becomes invertible in $\mathsf{H}_0(NB)$. When we restrict ourselves to the 0-simplex morphisms in A and to the degree zero morphisms in NA we obtain the same category, and similarly for B and NB. In fact, the degree zero component of the shuffle map ∇ (see Mac Lane [25, §VIII-8.8]) is the identity map. Therefore, x corresponds to an object in A, y corresponds to an object in B and v corresponds to a morphism $\tilde{v} : Fx \to y$ in $\mathcal{B}$ such that $v = N(\tilde{v})$. Thanks to Lemma 4.17, the morphism $\tilde{v}$ is invertible. By hypothesis F satisfies condition (k-F2), and so there exists a morphism $\tilde{u} : x \to x'$ in A, which becomes invertible in $\pi_0(A)$, such that $F(\tilde{u}) = (\tilde{v})$. By Lemma 4.17 the morphism $u := N(\tilde{u})$ becomes invertible in $\mathsf{H}_0(NA)$. Therefore, since $NF(u) = v$, we conclude that the dg functor NF satisfies condition $(\mathrm{F2})_{\geqslant 0}$.

Now, suppose that NF satisfies condition $(\mathrm{F2})_{\geqslant 0}$. Let x be an object in A and

$$v : Fx \longrightarrow y$$

a morphism in B, which becomes invertible in $\pi_0(B)$. By Lemma 4.17, we obtain a morphism in NB

$$N(v) : NFx \longrightarrow y,$$

which becomes invertible in $\mathsf{H}_0(NB)$. Since by hypothesis, NF satisfies condition $(\mathrm{F2})_{\geqslant 0}$, there exists a morphism $\tilde{u} : x \to x'$ in NA, which becomes invertible in $\mathsf{H}_0(NA)$, such that $NF(\tilde{u}) = N(v)$. As before, when we restrict ourselves to the 0-simplex morphisms in A and to the degree zero morphisms in NA we obtain the same category, and similarly for B and NB. Therefore, $\tilde{u}$ corresponds to a morphism $u : x \to x'$ in A such that $N(u) = \tilde{u}$. Thanks to Lemma 4.17, the morphism u is invertible in $\pi_0(A)$. Finally, since $F(u) = v$, we conclude that F satisfies condition (k-F2). The proof is then finished. □

In order to prove Theorem 4.16, we now verify the conditions of Theorem A.2, with respect to the adjunction (L, N) and the Quillen model structure of Theorem 3.10. Thanks to Propositions 4.18 and 4.19, a simplicial k-linear functor $F : A \to B$ is a weak equivalence if the dg functor $NF : NA \to NB$ is a weak equivalence in the model structure of Theorem 3.10. Similarly, a simplicial k-linear functor F is a fibration if the dg functor NF is a fibration in the model structure of Theorem 3.10. By construction, the model structure on $\mathsf{dgcat}_{\geqslant 0}$ is cofibrantly generated and by Lemma 3.21 every object in $\mathsf{dgcat}_{\geqslant 0}$ is fibrant. Since the functor N commutes with filtered colimits it remains to show the following:

– for each simplicial k-linear category A we have a factorization

$$\begin{tikzcd} A \arrow[rr, "\Delta"] \arrow[dr, "\gamma"', "\sim"] & & A \times A \\ & P(A), \arrow[ur, two heads, "\pi_0 \times \pi_1"'] & \end{tikzcd}$$

where γ is a weak equivalence and $\pi_0 \times \pi_1$ is a fibration in $s\mathbf{Mod}$-Cat.

This will be done in Proposition 4.27. We start by constructing a path object (see Definition 2.27) in $s\mathbf{Mod}$-Cat.

Definition 4.20. Let A be a simplicial k-linear category. The simplicial k-linear category $P(A)$ is defined as follows: its objects are the morphisms $f : x \to y$ in A which become invertible in $\pi_0(A)$ (see Definition 4.11). Given two objects $f : x \to y$ and $f' : x' \to y'$ in $P(A)$, the simplicial k-module of morphisms $P(A)(f, f')$ is the homotopy pullback of the following diagram in $s\mathbf{Mod}$

$$\begin{tikzcd} & A(y, y') \arrow[d, "f^*"] \\ A(x, x') \arrow[r, "f'_*"'] & A(x, y'). \end{tikzcd}$$

By this we mean the simplicial k-module (see Section 1.1)

$$A(x, x') \underset{A(x,y')}{\times} \underline{s\mathbf{Mod}}\big(k\Delta[1], A(x, y')\big) \underset{A(x,y')}{\times} A(y, y').$$

The simplexes in $A(x, x')$ and $A(y, y')$ are called *lateral morphisms* and the simplexes in $s\mathbf{Mod}(k\Delta[1], A(x, y'))$ are called *homotopies*. Under this notation, the composition operation

$$P(A)\big(f', f''\big) \wedge P(A)\big(f, f'\big) \longrightarrow P(A)\big(f, f''\big) \quad f, f', f'' \in P(A)$$

is given by:

– the composition of lateral morphisms (induced by the composition of A);
– the composition of homotopies, given by:

$$\begin{tikzcd} \underline{s\mathbf{Mod}}(k\Delta[1], A(x', x'')) \wedge A(x, x') \underset{\underline{s\mathbf{Mod}}(k\Delta[0], A(x, y''))}{\times} A(y', y'') \wedge \underline{s\mathbf{Mod}}(k\Delta[1], A(x, y')) \arrow[d, "\text{composition}"] \\ \underline{s\mathbf{Mod}}(k\Delta[1] \bigoplus_{k\Delta[0]} k\Delta[1], A(x, y'')) \arrow[d] \\ \underline{s\mathbf{Mod}}(k\Delta[1], A(x, y'')), \end{tikzcd}$$

where the last map is induced by the diagonal map in $k\Delta[1]$.

Remark 4.21. A 0-simplex morphism $\alpha : f \to f'$ in $P(A)$ is of the form $\alpha = (m_x, h, m_y)$, with $m_x : x \to x'$ and $m_y : y \to y'$ 0-simplex morphisms and h a 1-simplex morphism in $A(x, y')$, such that $d_0(h) = m_y \circ f$ and $d_1(h) = f' \circ m_x$.

Note that we have a commutative diagram in s**Mod**-Cat

$$\begin{tikzcd} A \arrow[rr, "\Delta"] \arrow[dr, "\gamma"'] & & A \times A \\ & P(A), \arrow[ur, "\pi_0 \times \pi_1"'] & \end{tikzcd} \tag{4.10}$$

where γ sends an object x in A to the identity $(\mathbf{1}_x : x = x)$ and $\pi_0 \times \pi_1$ sends an object $f : x \to y$ in $P(A)$ to (x, y). If we apply the normalization functor (see Eq. (4.9))

$$N : s\mathbf{Mod}\text{-Cat} \longrightarrow \mathsf{dgcat}_{\geqslant 0}$$

to the above commutative diagram (4.10) and then apply Proposition 3.27 to the connective dg category NA, we obtain two factorizations:

$$\begin{tikzcd} NA \arrow[rr, "\Delta"] \arrow[dr] \arrow[ddr, bend right, "\tau_{\geqslant 0}(\gamma)"'] & & NA \times NA \\ & NP(A) \arrow[ur] & \\ & \tau_{\geqslant 0} P(i(NA)). \arrow[uur, bend right, "\tau_{\geqslant 0}(\pi_0) \times \tau_{\geqslant 0}(\pi_1)"'] & \end{tikzcd}$$

Thanks to Proposition 3.27, $\tau_{\geqslant 0} P(i(NA))$ is a path object for NA in $\mathsf{dgcat}_{\geqslant 0}$. We will show in Proposition 4.27 that $NP(A)$ is also a path object for NA in $\mathsf{dgcat}_{\geqslant 0}$.

Lemma 4.22. *Let $A, B \in s$**Mod**. The Eilenberg–Mac Lane shuffle map ∇ (see Mac Lane* [25, §VIII-8.8]) *induces a surjective homotopy equivalence*

$$\nabla^\sharp : N\big(\underline{s\mathbf{Mod}}(A, B)\big) \longrightarrow \underline{Ch_{\geqslant 0}(k)}(NA, NB),$$

which has a section induced by the Alexander–Whitney map.

Proof. Note first that if (L, R) and (L', R') are adjoint pairs of functors, a natural transformation $\zeta : L \to L'$ induces a natural transformation $\zeta^\sharp : R' \to R$ which is an equivalence if and only if ζ is also.

Fixing a non-negatively graded complex $NA \in Ch_{\geqslant 0}(k)$, we define the functors $L, L' : Ch_{\geqslant 0}(k) \to Ch_{\geqslant 0}(k)$ by

$$L(C) := C \otimes NA \quad \text{and} \quad L'(C) := N(\Gamma C \wedge A).$$

Using the Dold–Kan equivalence [15, §III-2.3] in the case of L', we see that these functors have right adjoints

$$R(C) = \underline{Ch_{\geqslant 0}(k)}(NA, C), \qquad R'(C) = N\big(\underline{s\mathbf{Mod}}(A, \Gamma C)\big).$$

The shuffle map determines a natural inclusion $\nabla : L \to L'$ which has a right inverse given by the Alexander–Whitney map AW; see Schwede and Shipley [28, Theorem 2.7]. It follows that $\nabla^\sharp : R' \to R$ is a natural surjection with a section given by $AW^\sharp$. Since ∇ is a homotopy equivalence, it is enough to show that the functors L, L', R, R' send chain homotopic maps to homotopic maps. The adjunctions (L, R) and (L', R') will then induce adjunctions on the homotopy category $Ho(Ch_{\geqslant 0}(k))$ and $\nabla : L \to L'$ will be a natural isomorphism between endofunctors of $Ho(Ch_{\geqslant 0}(k))$.

The functors L and R preserve the chain homotopy relation. For the same reason, L' and R' preserve the relation on $\mathsf{Hom}_{Ch_{\geqslant 0}(k)}(C, D)$ defined by the cylinder object:

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & N(\Gamma C \wedge k\Delta[1]) \arrow[d] & C \arrow[l] \arrow[dl] \\ & C. & \end{tikzcd}$$

Since the Alexander–Whitney and shuffle maps give maps between this cylinder object and the usual one, this relation is the usual chain homotopy relation. This finishes the proof. $\square$

We now define a map ϕ relating the $Ch_{\geqslant 0}(k)$-graphs (see Notation 4.4) associated to the connective dg categories $NP(A)$ and $\tau_{\geqslant 0} P(i(NA))$. Observe that:

– by Proposition 3.27, $NP(A)$ and $\tau_{\geqslant 0}P(i(NA))$ have the same objects and
– for each pair of objects $(f : x \to y)$, $(f' : x' \to y')$ in $NP(A)$, the map of Lemma 4.22 (with $A = k\Delta[1]$) induces a surjective quasi-isomorphism:

$$\begin{array}{c} NA(x,x') \underset{NA(x,y')}{\times} Ns\underline{\mathbf{Mod}}(k\Delta[1], A(x,y')) \underset{NA(x,y')}{\times} NA(y,y') \\ \sim \Big\downarrow \mathbf{1}\times\nabla^{\sharp}\times\mathbf{1} =: \phi_{f,f'} \\ NA(x,x') \underset{NA(x,y')}{\times} \underline{Ch_{\geqslant 0}(k)}(Nk\Delta[1], NA(x,y')) \underset{NA(x,y')}{\times} NA(y,y'). \end{array}$$

Notation 4.23. We denote by

$$\phi : NP(A) \dashrightarrow \tau_{\geqslant 0}P(i(NA))$$

the map of $Ch_{\geqslant 0}(k)$-graphs which is the identity on objects and $\phi_{f,f'}$ on the complexes of morphisms.

Remark 4.24. Thanks to Remark 4.21 (and Definition 4.20), the map ϕ preserve identities and composition of degree zero morphisms.

We now establish a 'homotopy equivalence lifting property' of ϕ.

Lemma 4.25. *Let α be a degree zero morphism in $\tau_{\geqslant 0}P(i(NA))$ which becomes invertible in $\mathsf{H}_0(\tau_{\geqslant 0}P(i(NA)))$. Then there exists a degree zero morphism $\overline{\alpha}$ in $NP(A)$, which becomes invertible in $\mathsf{H}_0(NP(A))$, such that $\phi(\overline{\alpha}) = \alpha$.*

Proof. Let

$$\alpha : (f : x \to y) \longrightarrow \bigl(f' : x' \to y'\bigr)$$

be a degree zero morphism in $\tau_{\geqslant 0}P(i(NA))$. Notice that α is of the form (m_x, h, m_y), with $m_x : x \to x'$ and $m_y : y \to y'$ degree zero morphisms in NA and $h : x \to y$ a degree 1 morphism in NA. By definition of $P(A)$, we can choose a representative $\overline{h} \in A(x,y)_1$ of h. We obtain then a degree zero morphism $\overline{\alpha} = (m_x, \overline{h}, m_y)$ in $NP(A)$ such that

$$\phi_{f,f'} : NP(A)\bigl(f, f'\bigr) \longrightarrow \tau_{\geqslant 0}P(NA)\bigl(f, f'\bigr),$$
$$\overline{\alpha} = (m_x, \overline{h}, m_y) \longmapsto (m_x, h, m_y).$$

Now suppose that α is invertible in $\mathsf{H}_0(\tau_{\geqslant 0}P(i(NA)))$. Then there exist morphisms β of degree 0 and r_1 and r_2 of degree 1 such that $d(r_1) = \beta \circ \alpha - \mathbf{1}$ and $d(r_2) = \alpha \circ \beta - \mathbf{1}$. As above, we can lift β to a morphism $\overline{\beta}$ in $NP(A)$. Since the map ϕ preserves identities and composition of degree zero morphisms, it maps $\overline{\alpha} \circ \overline{\beta}$ to $\alpha \circ \beta$ and $\overline{\beta} \circ \overline{\alpha}$ to $\beta \circ \alpha$. Finally, since the maps $\phi_{f,f'}$ are surjective quasi-isomorphisms, we apply Lemma 4.26 (with $R = \phi_{f,f}$, $n = 0$, $a = \overline{\beta} \circ \overline{\alpha} - \mathbf{1}$, and $b = r_1$), to obtain a morphism r'_1 such that $d(r'_1) = \overline{\beta} \circ \overline{\alpha} - \mathbf{1}$. By applying Lemma 4.26 (with $R = \phi_{f',f'}$, $n = 0$, $a = \overline{\alpha} \circ \overline{\beta} - \mathbf{1}$, and $b = r_2$), we obtain a morphism r'_2 such that $d(r'_2) = \overline{\alpha} \circ \overline{\beta} - \mathbf{1}$. This shows that $\overline{\alpha}$ invertible in $\mathsf{H}_0(NP(A))$. □

Lemma 4.26. *Let $R : M_\bullet \to N_\bullet$ be a surjective quasi-isomorphism of non-negatively graded chain complexes. Given a non-negative integer n, and elements $a \in M_n$ and $b \in N_{n+1}$ such that $R(a) = d(b)$ and $d(a) = 0$, there exists an element $b' \in M_{n+1}$ such that $R(b') = b$ and $d(b') = a$.*

Proof. Since R is a surjective map, we have a short exact sequence of complexes

$$0 \to K \hookrightarrow M_\bullet \xrightarrow{R} N_\bullet \to 0, \tag{4.11}$$

where K denotes the kernel of R. Moreover, since R is a quasi-isomorphism the homology of K is trivial. Choose first a $w \in M_{n+1}$ such that $R(w) = b$. Consider the element $d(w) - a$. Note that $d(d(w) - a) = 0$ and $R(d(w) - a) = 0$. The above short exact sequence (4.11) and the triviality of the homology of K imply that there exists a $v \in M_{n+1}$ such that $d(v) = d(w) - a$ and $R(v) = 0$. Finally, let $b' := w - v$. Then $R(b') = b$ and $d(b') = a$ as required. □

Proposition 4.27. *Consider the following commutative diagram in* $\mathsf{dgcat}_{\geqslant 0}$

$$\begin{tikzcd} NA \arrow[rr, "\Delta"] \arrow[dr, "N(\gamma)"'] & & NA \times NA \\ & NP(A), \arrow[ur, "N(\pi_0)\times N(\pi_1)"'] & \end{tikzcd}$$

obtained by applying the normalization functor $N : s\mathbf{Mod} \to \mathsf{dgcat}_{\geqslant 0}$ to the diagram (4.10)*. Then, the dg functor $N(\gamma)$ is a quasi-equivalence* (*see Definition* 3.5) *and $N(\pi_0) \times N(\pi_1)$ is a fibration* (*see Definition* 3.8)*.*

Proof. We start by showing that $N(\gamma)$ is a quasi-equivalence. By construction, the dg functor $N(\gamma)$ satisfies condition $(\mathrm{QE1})_{\geqslant 0}$. We now show that $N(\gamma)$ satisfies condition $(\mathrm{QE2})_{\geqslant 0}$. Let f be an object in $NP(A)$. The dg categories $NP(A)$ and $\tau_{\geqslant 0}P(i(NA))$ have the same objects and so we can consider f as an object in $\tau_{\geqslant 0}P(i(NA))$. Since by Proposition 3.27, the dg functor

$$\tau_{\geqslant 0}(\gamma) : NA \longrightarrow \tau_{\geqslant 0}P\big(i(NA)\big)$$

is a quasi-equivalence, there exists an object x in NA and a homotopy equivalence α in $\tau_{\geqslant 0}P(i(NA))$ between $\gamma(x) = (\mathbf{1}_x : x = x)$ and f. Thanks to Proposition 4.25, we can lift α to a homotopy equivalence $\overline{\alpha}$ in $NP(A)$. This implies that the dg functor

$$N(\gamma) : NA \longrightarrow NP(A)$$

satisfies condition $(\mathrm{QE2})_{\geqslant 0}$. In sum, $N(\gamma)$ is a quasi-equivalence.

We now show that $N(\pi_0) \times N(\pi_1)$ is a fibration. By construction, the dg functor $N(\pi_0) \times N(\pi_1)$ satisfies condition $(\mathrm{F1})_{\geqslant 0}$. We now show that it satisfies condition $(\mathrm{F2})_{\geqslant 0}$. Let $f : x \to y$ be an object in $NP(A)$ and $\gamma : (x, y) \to (x', y')$ a homotopy equivalence in $NA \times NA$. Thanks to Proposition 3.27, the dg functor

$$\tau_{\geqslant 0}(\pi_0) \times \tau_{\geqslant 0}(\pi_1) : \tau_{\geqslant 0}P\big(i(NA)\big) \longrightarrow NA \times NA$$

is a fibration, and so there is a homotopy equivalence $\alpha : f \to f'$ in $\tau_{\geqslant 0}P(i(NA))$ such that $\tau_{\geqslant 0}(P)(\alpha) = \gamma$. By Lemma 4.25, we can lift α to a homotopy equivalence $\overline{\alpha} : f \to f'$ in $N(PA)$ such that $N(\pi_0) \times N(\pi_1)(\overline{\alpha}) = \gamma$. This finishes the proof. □

Remark 4.28. Thanks to Proposition 4.27, the proof of Theorem 4.16 is finished. Note that, since every object in $\mathsf{dgcat}_{\geqslant 0}$ is fibrant (see Lemma 3.21), all simplicial k-linear categories are fibrant with respect to the model structure of Theorem 4.16.

4.5. Quillen equivalence

Lemma 4.29. *Let $\mathcal{A}$ be a connective dg category. We denote by I its set of objects. Suppose that $\mathcal{A}$ is cofibrant in $\mathsf{dgcat}_{\geqslant 0}$* (*see Theorem* 3.10)*. Then $\mathcal{A}$ is also cofibrant in the category $Ch_{\geqslant 0}(k)$-Cat_I* (*see Remark* 4.6)*.*

Proof. Given a solid diagram in $Ch_{\geqslant 0}(k)$-Cat_I

$$\begin{tikzcd} & \mathcal{B} \arrow[d, "P", "\sim"'] \\ \mathcal{A} \arrow[r, "F"'] \arrow[ur, dashed, "\overline{F}"] & \mathcal{C}, \end{tikzcd}$$

with P a trivial fibration, we need to construct a morphism $\overline{F}$ making the above diagram commute. In the proof of Theorem 3.10 we have shown (see Lemma 3.17) that the trivial fibrations in $\mathsf{dgcat}_{\geqslant 0}$ are the dg functors of the class $\mathbf{Surj}_{\geqslant 0}$ (see Definition 3.13). Since the dg functor P induces the identity map on objects, and by Lemma 3.16, the morphisms in $Ch_{\geqslant 0}(k)$

$$P(x, y) : \mathcal{B}(x, y) \longrightarrow \mathcal{C}(x, y) \quad x, y \in I$$

are surjective quasi-isomorphisms, we conclude that P belongs to $\mathbf{Surj}_{\geqslant 0}$, By hypothesis, the dg category $\mathcal{A}$ is cofibrant in dgcat, and so we obtain a dg functor $\overline{F}$ such that $P \circ \overline{F} = F$. Finally, since F and P induce the identity map on objects, so does the dg functor $\overline{F}$. This shows that $\overline{F}$ belongs to $Ch_{\geqslant 0}(k)$-Cat_I, and so the proof is finished. □

Theorem 4.30. *The Quillen adjunction* (*see Theorem* 4.16)

$$\begin{tikzcd} s\mathbf{Mod}\text{-}\mathrm{Cat} \arrow[d, shift left, "N"] \\ \mathsf{dgcat}_{\geqslant 0} \arrow[u, shift left, "L"] \end{tikzcd}$$

is a Quillen equivalence; *see Hirschhorn* [13, Definition 8.5.20].

Remark 4.31. The proof of Theorem 4.30 is based on the fact that the functors L and N do not alter the sets of objects, i.e. given a connective dg category $\mathcal{A}$ and a simplicial k-linear category B, the simplicial k-linear category $L(\mathcal{A})$ has the same objects as $\mathcal{A}$ and the connective dg category $N(B)$ has the same objects as B. These facts allow us to reduce the problem to the fixed set of objects case, where we can use Schwede–Shipley's previous results (see [28, §6]).

Proof. Let $\mathcal{A} \in \mathsf{dgcat}_{\geqslant 0}$ be a cofibrant dg category and B a simplicial k-linear category. Recall from Remark 4.28 that every object in $s\mathbf{Mod}$-Cat is fibrant. We need to show that a simplicial k-linear functor

$$F : L(\mathcal{A}) \longrightarrow B$$

is a weak equivalence in $s\mathbf{Mod}$-Cat (see Definition 4.13) if and only if the corresponding dg functor

$$F^\sharp : \mathcal{A} \longrightarrow NB$$

is a quasi-equivalence in $\mathsf{dgcat}_{\geqslant 0}$ (see Definition 3.5). We have the following commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F^\sharp"] \arrow[d, "\eta"'] & NB \\ NL(\mathcal{A}), \arrow[ur, "NF"'] & \end{tikzcd}$$

where η is the counit of the adjunction (L, N). Since by Proposition 4.18, F is a weak equivalence in $s\mathbf{Mod}$-Cat if and only if NF is a quasi-equivalence, it is enough to show that η is a quasi-equivalence in $\mathsf{dgcat}_{\geqslant 0}$. Note that the dg functor η induces the identity map on objects. If we denote by I the set of objects of $\mathcal{A}$, it remains then to show that the morphisms in $Ch_{\geqslant 0}(k)$

$$\eta(x, y) : \mathcal{A}(x, y) \longrightarrow NL(\mathcal{A})(x, y) \quad x, y \in I$$

are quasi-isomorphisms. Since $\mathcal{A}$ is cofibrant in $\mathsf{dgcat}_{\geqslant 0}$, Lemma 4.29 implies that $\mathcal{A}$ is also cofibrant in the category $Ch_{\geqslant 0}(k)\text{-Cat}_I$ (see Remark 4.6). Therefore by Schwede and Shipley [28, Proposition 6.4] the morphism $\Gamma(\mathcal{A}) \to L_I(\mathcal{A})$ in $s\mathbf{Mod}\text{-Gr}_I$, adjoint to the unit morphism $\mathcal{A} \to N_I L_I(\mathcal{A})$ in $Ch_{\geqslant 0}(k)\text{-Gr}_I$ (see adjunctions (4.7), (4.5) and (4.8)), is a weak equivalence; see Remark 4.7. This implies that the morphisms in $s\mathbf{Mod}$

$$\Gamma(\mathcal{A})(x, y) \longrightarrow L_I(\mathcal{A})(x, y) \quad x, y \in I$$

are weak equivalences (see Section 1.1). Finally, by the Dold–Kan equivalence [15, §III-2.3], we obtain quasi-isomorphisms

$$\eta(x, y) : \mathcal{A}(x, y) \xrightarrow{\sim} N\big(\Gamma(\mathcal{A})(x, y)\big) \xrightarrow{\sim} N\big(L_I(\mathcal{A})(x, y)\big) = NL(\mathcal{A})(x, y) \quad x, y \in I,$$

and so the proof is finished. □

5. Simplicial categories

In this section, we will construct a Quillen adjunction between simplicial k-linear categories and simplicial categories; see Theorem 5.8.

Recall from Section 1.1 that the category of simplicial sets is denoted by $s\mathbf{Set}$.

Definition 5.1. A *simplicial category* is a $s\mathbf{Set}$-category (see Borceaux [7, Definition 6.2.1]) and a *simplicial functor* is a $s\mathbf{Set}$-functor; see [7, Definition 6.2.3].

Remark 5.2. Note that Definition 5.1 is no more than a combination of Definitions 2.1 and 2.3 in which $Ch(k)$ was replaced with $s\mathbf{Set}$.

Notation 5.3. We denote by $s\mathbf{Set}$-Cat the category of simplicial categories.

Definition 5.4. Let A be a simplicial category. The category $\pi_0(A)$ has the same objects as A and morphisms given by $\pi_0(A)(x, y) := \pi_0(A(x, y))$, where $\pi_0(A(x, y))$ is the set of connected components of the simplicial set $A(x, y)$. We obtain a well-defined functor

$$\pi_0(-) : s\mathbf{Set}\text{-Cat} \longrightarrow \text{Cat},$$

with values in the category of categories.

Definition 5.5. A simplicial functor $F : A \to B$ is a *Dwyer–Kan equivalence* if:

(DK1) for all objects $x, y \in A$, the morphism in $s\mathbf{Set}$

$$F(x, y) : A(x, y) \longrightarrow B(Fx, Fy)$$

is a weak equivalence (see Section 1.1) and

(DK2) the induced functor

$$\pi_0(F) : \pi_0(A) \longrightarrow \pi_0(B)$$

is an equivalence of categories.

Definition 5.6. A simplicial functor $F : A \to B$ is a *fibration* if:

(SF1) for all objects $x, y \in A$, the morphism in $s\mathbf{Set}$

$$F(x, y) : A(x, y) \longrightarrow B(Fx, Fy)$$

is a Kan-fibration (see Section 1.1) and

(SF2) given an object x in A and a morphism $v : Fx \to y$ in B, which becomes invertible in $\pi_0(B)$, there exists a morphism $u : x \to x'$ in A, which becomes invertible in $\pi_0(A)$, such that $Fu = v$.

Theorem 5.7. *(Bergner [2, Theorem 1.1]) The category s***Set***-Cat is a Quillen model category with weak equivalences the Dwyer–Kan equivalences and fibrations those of Definition* 5.6. *The cofibrations are the simplicial functors which have the left lifting property with respect to the simplicial functors which are simultaneously fibrations and weak equivalences.*

Recall from [15, §III], the adjunction between simplicial k-modules and simplicial sets

$$\begin{array}{c} s\mathbf{Mod} \\ {\scriptstyle k(-)}\big\uparrow \;\big\downarrow{\scriptstyle U} \\ s\mathbf{Set}, \end{array}$$

where U is the forgetful functor and $k(-)$ the k-linearization functor. The functor $k(-)$ is strong monoidal (see Section 1.2), and so we obtain an adjunction

$$\begin{array}{c} s\mathbf{Mod}\text{-Cat} \\ {\scriptstyle k(-)}\big\uparrow \;\big\downarrow{\scriptstyle U} \\ s\mathbf{Set}\text{-Cat}. \end{array}$$

Theorem 5.8. *The preceding adjunction* $(k(-), U)$ *is a Quillen adjunction* (*see Hirschhorn* [13, Definition 8.5.2]), *with respect to the model structures of Theorems* 4.16 *and* 5.7.

Proof. Note first that conditions (WE1) and (WE2) of Definition 4.13 correspond to conditions (DK1) and (DK2) of Definition 5.5. This shows that the functor

$$U : s\mathbf{Mod}\text{-Cat} \longrightarrow s\mathbf{Set}\text{-Cat}$$

preserves weak equivalences. Let us now show that it also preserves fibrations. Given a fibration $F : A \to B$ in $s\mathbf{Mod}$-Cat (see Definition 4.14), Proposition 4.19 implies that $NF : NA \to NB$ is a fibration in $\mathsf{dgcat}_{\geqslant 0}$. We need to show that the simplicial functor $UF : UA \to UB$ is a fibration; see Definition 5.6. Since $NF : NA \to NB$ is a fibration in $\mathsf{dgcat}_{\geqslant 0}$, the morphisms in $Ch_{\geqslant 0}(k)$

$$NF(x, y) : NA(x, y) \longrightarrow NB(Fx, Fy) \quad x, y \in NA$$

are surjective in positive degrees. Thanks to Goerss and Jardine [15, Lemma 2.11(2)-III], we conclude that the morphisms

$$UF(x, y) : UA(x, y) \longrightarrow UB(UFx, UFy) \quad x, y \in A$$

are fibrations in $s\mathbf{Set}$, and so the simplicial functor UF satisfies condition (SF1).

We now show that UF satisfies also condition (SF2). Let x be an object in UA and $v : UFx \to y$ a morphism in UB which becomes invertible in $\pi_0(UB)$. Note that we can consider x as an object in A, y as an object in B and v as a morphism $v : Fx \to y$ in B, which becomes invertible in $\pi_0(B)$. Therefore by Lemma 4.17, $N(v) : NFx \to y$ is invertible in $\mathsf{H}_0(NB)$. Since $NF : NA \to NB$ is a fibration, there exists a morphism $\tilde{u} : x \to x'$, which becomes invertible in $\mathsf{H}_0(NA)$, such that $NF(\tilde{u}) = N(v)$. When we restrict ourselves to the 0-simplex morphisms in A and to the degree zero morphisms in NA we obtain the same category, and similarly for B and NB. In fact, the degree zero component of the shuffle map ∇ (see Mac Lane [25, §VIII-8.8]) is the identity map. Therefore $\tilde{u}$ corresponds to a morphism $u : x \to x'$ in A such that $F(u) = v$. By Lemma 4.17, u is invertible in $\pi_0(A)$. In conclusion, we obtain a morphism $U(u) = u : x \to x'$ which becomes invertible in $\pi_0(UA)$, such that $UF(u) = v$. This shows that $UF : UA \to UB$ satisfies condition (SF2), and so the proof is finished. $\square$

6. Global picture

We now sum up our three-step zig-zag of Quillen adjunctions relating the homotopy theories of differential graded and simplicial categories (see Proposition 3.25, and Theorems 4.16 and 5.8):

$$\begin{tikzcd} s\mathbf{Set}\text{-Cat} \arrow[d, shift right, "k(-)"'] \\ s\mathbf{Mod}\text{-Cat} \arrow[u, shift right, "U"'] \arrow[d, shift left, "N"] \\ \mathsf{dgcat}_{\geqslant 0} \arrow[u, shift left, "L"] \arrow[d, hook, shift right, "i"'] \\ \mathsf{dgcat}. \arrow[u, shift right, "\tau_{\geqslant 0}"'] \end{tikzcd}$$

Thanks to Theorem 4.30 the Quillen adjunction (L, N) is a Quillen equivalence, and so both (derived) functors between the homotopy categories

$$\mathbb{L}L : Ho(\mathsf{dgcat}) \longrightarrow Ho(s\mathbf{Mod}\text{-Cat}) \qquad \mathbb{R}N : Ho(s\mathbf{Mod}\text{-Cat}) \longrightarrow Ho(\mathsf{dgcat}_{\geqslant 0})$$

commute with homotopy limits and colimits. This implies that the composed functor

$$\Phi := U \circ \mathbb{L}L \circ \tau_{\geqslant 0} : Ho(\mathsf{dgcat}) \longrightarrow Ho(s\mathbf{Set}\text{-Cat})$$

preserves homotopy limits and the composed functor

$$\Psi := i \circ \mathbb{R}N \circ \mathbb{L}k(-) : Ho(s\mathbf{Set}\text{-Cat}) \longrightarrow Ho(\mathsf{dgcat})$$

preserves homotopy colimits. A consequence of these results is the following:

Corollary 6.1.

(1) *Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor and b an object of $\mathcal{B}$. We denote by $\mathbf{HFib}(F)/b$ the homotopy fiber of F over b. Then, we have an isomorphism*

$$\Phi\big(\mathbf{HFib}(F)/b\big) \xrightarrow{\sim} \mathbf{HFib}\big(\Phi(F)\big)/b$$

in the homotopy category $Ho(s\mathbf{Set}\text{-Cat})$.

(2) *Let $G : C \to D$ be a simplicial functor. We denote by $\mathbf{HCof}(G)$ the homotopy cofiber of G, i.e. the homotopy cobase change along the map to the terminal object. Then, we have an isomorphism*

$$\mathbf{HCof}\big(\Psi(G)\big) \xrightarrow{\sim} \Psi\big(\mathbf{HCof}(G)\big)$$

in the homotopy category $Ho(\mathsf{dgcat})$.

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Appendix A. Homotopical algebra tools

In this appendix we will state two classical results (used in the paper) which allow us to construct Quillen model structures [21]. The first result is Quillen's path object argument; see Quillen [21, §II-4], Schwede and Shipley [29, Lemma 2.3] or Berger and Moerdijk [1, §2.6]. Let

$$\begin{tikzcd} \mathcal{M} \arrow[d, shift left, "U"] \\ \mathcal{N} \arrow[u, shift left, "F"] \end{tikzcd}$$

be an adjunction between complete and cocomplete categories. Assume that $\mathcal{N}$ carries a cofibrantly generated Quillen model structure and that U preserves filtered colimits.

Definition A.1. A morphism $f : A \to B$ in $\mathcal{M}$ is a *weak equivalence* if the morphism $U(f)$ is a weak equivalence in $\mathcal{N}$. Similarly, a morphism f in $\mathcal{M}$ is a *fibration* if the morphism $U(f)$ is a fibration in $\mathcal{N}$. A *cofibration* in $\mathcal{M}$ is a morphism which has the left lifting property with respect to the morphisms which are simultaneously fibrations and weak equivalences.

Theorem A.2. *Suppose that for every object $A \in \mathcal{M}$, the unique map $A \to *$ (where $*$ denotes the terminal object in $\mathcal{M}$) is a fibration and that we have a factorization in $\mathcal{M}$*

$$\begin{tikzcd} A \arrow[rr, "\Delta"] \arrow[dr, "\sim", "\gamma"'] & & A \times A \\ & P(A), \arrow[ur, "\pi_0 \times \pi_1"'] & \end{tikzcd}$$

with γ a weak equivalence and $\pi_0 \times \pi_1$ a fibration in $\mathcal{M}$. Then, the category $\mathcal{M}$ endowed with the notions of weak equivalence, fibration, and cofibration of Definition A.1 *is a cofibrantly generated Quillen model structure. Moreover, the adjunction (F, U) is a Quillen adjunction.*

The second result is the following standard recognition theorem; see Hovey [14, Theorem 2.1.19].

Theorem A.3. *Let $\mathcal{M}$ be a complete and cocomplete category, W a class of maps in $\mathcal{M}$, and I and J sets of maps in $\mathcal{M}$ such that*:

(1) *The class W satisfies the two out of three axioms and is stable under retracts.*
(2) *The domains of the elements of I are small relative to I-*cell.
(3) *The domains of the elements of J are small relative to J-*cell.
(4) J-cell $\subseteq W \cap I$-cof.
(5) I-inj $\subseteq W \cap J$-inj.
(6) $W \cap I$-cof $\subseteq J$-cof *or* $W \cap J$-inj $\subseteq I$-inj.

Then there is a cofibrantly generated model category structure on $\mathcal{M}$ in which W is the class of weak equivalences, I is a set of generating cofibrations, and J is a set of generating trivial cofibrations.

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